105. A Characterization of Smooth Banach Spaces

By Bruce CALVERT*) and Simeon REICH**)

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1981)

Our main purpose in this note is to show how (nonlinear) accretive operators can be used to characterize smooth (reflexive) Banach spaces. More precisely, we show that if a Banach space E is not smooth, then there is accretive $A\subset E\times E$ that satisfies the range condition such that cl $(R(A))$ does not have the minimum property (see the definitions below). On the other hand, it is also true that if a reflexive space E is smooth and $A\subset E\times E$ is an accretive operator that satisfies the range condition, then cl $(R(A))$ has the minimum property. Consequently, a reflexive Banach space E is smooth if and only if cl $(R(A))$ has the minimum property for all accretive $A\subset E\times E$ that satisfy the range condition (Theorem 1). In fact, the same result is true if A is restricted to be of the form $I-T$, where T is nonexpansive. This provides an answer to a question of Pazy [3]. In addition, we characterize (finite-dimensional) smooth Banach spaces by using an invariance criterion for nonlinear semigroups (Theorem 2).

Let E be a real Banach space, and let I denote the identity operator. Recall that a subset A of $E\times E$ with domain $D(A)$ and range $R(A)$ is said to be *accretive* if $|x_1-x_2|\le|x_1-x_2+r(y_1-y_2)|$ for all $[x_i, y_i]\in A$, $i=1,2$, and $r>0$. The resolvent $J_r: R(I+rA)\to D(A)$ of A is defined by $J_r=(I+rA)^{-1}$. We denote the closure of a subset D of E by cl (D), its closed convex hull by clco (D) and its distance from a point x in E by $d(x, D)$. We shall say that A satisfies the *range condition* if $R(I+rA)\supset$ cl $(D(A))$ for all $r>0$. In this case, $-A$ generates a nonexpansive nonlinear semigroup $S:[0,\infty)\times$ cl $(D(A))\to$ cl $(D(A))$ by the exponential formula: $S(t)x=\lim_{n\to\infty}(I+(t/n)A)^{-n}x$. A closed subset D of E is said to have the *minimum property* [3] if $d(O, \text{clco}(D))=d(O, D)$.

Recall that the norm of E is said to be Gâteaux differentiable (and E is said to be *smooth*) if $\lim_{t\to 0}(|x+ty|-|x|)/t$ exists for each x and y in $U=\{x\in E:|x|=1\}$. The duality map from E into the family of nonempty subsets of its dual E^* is defined by $J(x)=\{x^*\in E^*:(x, x^*)=|x|^2=|x^*|^2\}$. It is single-valued if and only if E is smooth. An

*) Department of Mathematics, The University of Auckland, Auckland, New Zealand.

**) Department of Mathematics, The University of Southern California, Los Angeles, California 90007.

operator $A \subset E \times E$ is accretive if and only if for each $x_i \in D(A)$ and each $y_i \in Ax_i$, $i=1, 2$, there exists $j \in J(x_1-x_2)$ such that $(y_1-y_2, j) \geq 0$.

Theorem 1. *A reflexive Banach space E is smooth if and only if* cl $(R(A))$ *has the minimum property for all accretive* $A \subset E \times E$ *that satisfy the range condition.*

Proof. Suppose that E is not smooth. Then there are points x and y on the unit sphere of E such that $\lim_{t\to 0}(|x+ty|-|x|)/t$ does not exist. Let F be the plane determined by x, y and the origin. F is a two-dimensional space that is not smooth. We may assume that $(0, c)$ is a point of norm 1 where the norm of F is not differentiable. Let $t_1<0<t_2$, and let $g: [t_1, t_2] \to R$ be a real-valued function such that the slope of each one of the chords of its graph equals a tangential slope at $(0, c)$. We now define a set $A \subset F \times F$ by

$$A(a, b)=\begin{cases}(t_1, g(t_1)) & \text{if } a<0\\ \{(t, g(t)): t_1 \leq t \leq t_2\} & \text{if } a=0\\ (t_2, g(t_2)) & \text{if } a>0.\end{cases}$$

It is not difficult to check that A is accretive. It is, in fact, m-accretive (that is, $R(I+rA)=F$ for all positive r):

$$J_r(a, b)=\begin{cases}(a-rt_1, b-rg(t_1)) & \text{if } a<rt_1\\ (0, \mathrm{b}-rg(a/r)) & \text{if } rt_1 \leq a \leq rt_2\\ (a-rt_2, b-rg(t_2)) & \text{if } a>rt_2\end{cases}$$

for all $(a, b) \in F$ and $r>0$. If one chord of the graph of g passes through the origin, although $g(0) \neq 0$ (this is possible because there is more than one tangent at $(0, c)$), then we obtain $d(O, R(A))=|(O, g(0))| >0=d(O, \text{clco}\,(R(A)))$. In other words, as a subset of $E \times E$, A is an accretive operator that satisfies the range condition but does not possess the minimum property.

Conversely, we assume now that E is a reflexive smooth Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and $d=d(O, R(A))$. Let x and y belong to cl $(D(A))$, and let $t>s>0$. Since $((1/s)(J_t y-J_s x)+((1/s)-(1/t))(x-J_t y)+(1/t)(x-y), j) \geq 0$ for some j in $J(J_s x-J_t y)$, it follows that $|J_s x-J_t y| \leq (1-(s/t))|x-J_t y| +(s/t)|x-y|$. This inequality, in turn, implies that $(x-J_s x, j) \geq |x-J_t y|(s/t)(|x-J_t y|-|x-y|)$ for all j in $J(x-J_t y)$. Hence $((x-J_s x)/s, j_t) \geq |x-J_t y|^2/t^2-|x-J_t y||x-y|/t^2$ for all $j_t \in J((x-J_t y)/t)$. Suppose that a subsequence of $\{j_t\}$ converges weakly to j and that a subsequence of $(x-J_s x)/s$ converges weakly to z as t an s tend to infinity. $((x-J_s)/s, j) \geq d^2$, $|j|=d$, $|z|=d$, and $(z, j)=d^2$. In other words, j belongs to $J(z)$. But the duality map J is single-valued because E is smooth. Consequently, the weak $\lim_{t\to\infty} J((x-J_t y)/t)$ exists and is independent of x and y. The result now follows by combining this fact with the proof of [5, Theorem 1].

Theorem 1 remains true if A is restricted to be of the form $I-T$, where T is nonexpansive. This fact provides an answer to a question of Pazy [3]. It can be seen by considering the Yosida approximation $A_1=I-J_1$ of A:

$$A_1(a, b)=\begin{cases}(t_1, g(t_1)) & \text{if } a<t_1\\ (a, g(a)) & \text{if } t_1\leq a\leq t_2\\ (t_2, g(t_2)) & \text{if } a>t_2.\end{cases}$$

It can also be shown [6] that in the second part of the proof of Theorem 1 it suffices to assume that E^* is strictly convex. The first part of the proof is related to an example in [2].

Theorem 2. *A finite-dimensional Banach space E is smooth if and only if for all accretive operators $A\subset E\times E$ that satisfy the range condition, any closed convex subset C of* cl $(D(A))$ *that is invariant under the semigroup S generated by $-A$ is also invariant under the resolvent J_r, $r>0$.*

Proof. If E is not smooth, we consider the operator A constructed in the first part of the proof of Theorem 1. The exponential formula shows that

$$S(t)(a, b)=\begin{cases}(a-t_1t, b-g(t_1)t) & \text{if } a<0,\ 0\leq t\leq a/t_1\\ (0, b-tg(0)) & \text{if } a=0,\ t\geq 0\\ (a-t_2t, b-g(t_2)t) & \text{if } a>0,\ 0\leq t\leq a/t_2.\end{cases}$$

We assume that $g(0)=0$ and let $(a, b)\in F$ with $a<0$. Then the half-line $y-b=(g(t_1)/t_1)(x-a)$, $x\leq 0$, is invariant under S. We claim that if g is not linear (this is possible because there is more than one tangent at $(0, c)$), then this half-line is not invariant under J_r. Indeed, if it were invariant, we would obtain for $rt_1\leq x\leq 0$,

$$y-rg\left(\frac{x}{r}\right)-b=\frac{g(t_1)}{t_1}(-a),\ \frac{g(t_1)}{t_1}(x-a)-rg\left(\frac{x}{r}\right)=\frac{g(t_1)}{t_1}(-a),$$

and $g(t_1)/t_1=g(s)/s$ for all $t_1\leq s\leq 0$. In other words, g would be linear for ≤ 0.

Conversely, we observe that if C is invariant under S, then it is also invariant under $(I+(r/t)(I-S(t)))^{-1}$. Let $x\in C$ and $y_t=(I+(r/t)(I-J_t))^{-1}x$. Since E is finite-dimensional and smooth, the uniqueness part of [4, Theorem 2.1] shows that $\lim_{t\to 0+} y_t=\lim_{t\to 0+}(I+(r/t)(I-S(t)))^{-1}x$ belongs to C. We also have

$$((x-J_rx)/r-(x-y_t)/r,\ J(J_rx-J_ty_t))\geq 0,$$
$$|J_rx-J_ty_t|^2\leq(y_t-J_ty_t,\ J(J_rx-J_ty_t)),$$

and $|J_rx-y_t|\leq(2t/r)\,|x-y_t|$. Hence $\lim_{t\to 0+} y_t=J_rx$ and the proof is complete.

The first part of the proof of Theorem 2 is related to an observation in [1]. In the second part of the proof it is sufficient to assume that E is reflexive with a uniformly Gâteaux differentiable norm.

References

[1] B. Calvert and C. Picard: Opérateurs accrétifs et φ-accrétifs dans un espace de Banach. Hiroshima Math. J., **8**, 11–30 (1978).

[2] M. G. Crandall and T. M. Liggett: Generation of semigroups of nonlinear transformations on general Banach spaces. Amer. J. Math., **93**, 265–298 (1971).

[3] A. Pazy: Asymptotic behavior of contractions in Hilbert space. Israel J. Math., **9**, 235–240 (1971).

[4] S. Reich: Product formulas, nonlinear semigroups, and accretive operators. J. Funct. Anal., **36**, 147–168 (1980).

[5] ——: A solution to a problem on the asymptotic behavior of nonexpansive mappings and semigroups. Proc. Japan Acad., **56A**, 85–87 (1980).

[6] ——: On the asymptotic behavior of nonlinear semigroups and the range of accretive operators, I, II. J. Math. Anal. Appl. (to appear).