101. *Calculus on Gaussian White Noise. III*

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In the previous parts of this series [11], [12], we have given a systematic treatment of calculus on Gaussian white noise, which is a reformulation of Hida's works [1], [2]. In this part we will show further relations between Hida's approach and ours. We will use the same notations and definitions as in Part I and Part II.

§ 8. **Multiple Wiener integrals.** Here we assume that the Borel measure ν on T has no atoms. Let $\mathcal{E}\subset E_0=L^2(T,\nu)\subset\mathcal{E}^*$ be a triplet as in § 5 of Part II, and let μ be the measure of Gaussian white noise on $\mathcal{E}^*$ with characteristic functional $\exp[-\|\xi\|_0^2/2]$. The multiple Wiener integral $I_n(F_n)$ of F_n in $L^2(T^n,\nu^n)$ is defined as follows:

First, $I_1(F_1)$ is the limit of $\langle x,\xi_k\rangle$ in $(L^2)=L^2(\mathcal{E}^*,\mu)$, where $\{\xi_k\}$ is any sequence in $\mathcal{E}$ with $\|\xi_k-F_1\|_0\to 0$, as $k\to\infty$. Specially, put $W(B)\equiv I_1(I_B)$, where I_B denotes the indicator function of a Borel set B with $\nu(B)<\infty$. Secondary, let $\alpha=\{B_j\}$ be a countable Borel partition of T with $\nu(B_j)<\infty$ and let α^n be the collection of all subsets of T^n of the form $C=B_{j(1)}\times B_{j(2)}\times\cdots\times B_{j(n)}$, $B_{j(k)}\in\alpha$, $B_{j(k)}\cap B_{j(m)}=\phi$ for $k\neq m$. For such a set C in α^n, define

$$I_n(I_C)\equiv\prod_{k=1}^n W(B_{j(k)}).$$

Define $I_n(G_n)\equiv\sum a_kI_n(I_{C_k})$ for $G_n=\sum a_kI_{C_k}$ with $C_k\in\alpha^n$. Then we can define $I_n(F_n)$ by

$$I_n(F_n)\equiv\underset{\alpha\uparrow}{\mathrm{l.i.m.}}\, I_n(F_n^\alpha),\qquad F_n^\alpha\equiv\sum\nu^{-1}(C)(F_n,I_C)I_C, \tag{8.1}$$

where $\alpha\uparrow$ means refinements.

Theorem 8.1. (i) *For $F_n\in L^2(T^n,\nu^n)$, put $\varphi(x)=I_n(F_n)$, then we have*

$$(\mathcal{S}\varphi)(\xi)=\int_{T^n}F_n(u_1,\cdots,u_n)\xi(u_1)\cdots\xi(u_n)d\nu^n(u_1,\cdots,u_n).$$

(ii) *For any ψ in (L^2), there exist $F_n\in\hat{L}^2(T^n,\nu^n)$, $n\geq 0$, such that $\psi(x)$ is decomposed into the following orthogonal sum;*

$$\psi(x)=\sum_{n=0}^\infty I_n(F_n)\quad\textit{and}\quad\|\psi\|_{(L^2)}^2=\sum_{n=0}^\infty n!\,\|F_n\|_{\hat{L}^2(T^n,\nu^n)}^2.$$

We now remark that the symmetric L^2-space $\hat{L}^2(T^n,\nu^n)$ is naturally identified with the symmetric tensor product space $E_0^{\hat{\otimes}n}$. By Theorems 6.3 and 6.5, we have

Theorem 8.2. *For $G_n \in E_0^{\hat{\otimes} n}$, $A^*(G_n)1 = I_n(G_n)$ holds in (L^2). Moreover, for $\varphi \in \mathcal{H}^{(p)}$, $p \geq 0$, there exists a Ξ in* $\exp[\hat{\otimes} E_p]$ *such that*

$$\varphi(x) = \sum_{n=0}^{\infty} A^*(\pi^n \Xi)1 = \sum_{n=0}^{\infty} I_n(\pi^n \Xi).$$

§9. Hida's approach. In this section, take $T = R$, the reals, and let ν be the ordinal Lebesgue measure on R. Let $\mathcal{E}$ be the space of rapidly decreasing functions. Put

$$D \equiv \left(1 + x^2 - \frac{d^2}{dx^2}\right) \quad \text{and} \quad (\xi, \zeta)_p \equiv (D^p \xi, D^p \zeta)_0, \quad p \geq 0.$$

Let E_p be the completion of $\mathcal{E}$ with respect to the norm $\|\cdot\|_p$ and let $E_{-p} = E_p^*$ be the dual of E_p. Then $\mathcal{E}$ is the projective limit of E_p, so we can apply previous discussions on the triplet $\mathcal{E} \subset E_0 \subset \mathcal{E}^*$. In particular, define $B(t)$ by

(9.1) $B(t) = W([0, t])$, for $t \geq 0$ and $= -W([t, 0])$, for $t < 0$.

This is the Brownian motion in Hida's articles. Hida's definition of $\dot{B}(t)$ by using $\mathcal{T}$-transform can be rewritten by using $\mathcal{S}$-transform and the following relations are obtained;

$$\text{(9.2)} \quad \dot{B}(t) = \mathcal{S}^{-1} \lim_{h \downarrow 0} \mathcal{S}\left(\frac{B(t+h) - B(t)}{h}\right) = \mathcal{S}^{-1} \lim_{h \downarrow 0} \frac{1}{h} \int_0^h \xi(t+u)\,du$$
$$= \mathcal{S}^{-1}(\xi(t)) = \partial_t^* 1.$$

Hida's definition of the multiplication by $\dot{B}(t)$ is equivalent to

$$\text{(9.3)} \quad \dot{B}(t)\psi(x) = \mathcal{S}^{-1} \lim_{h \downarrow 0} \mathcal{S}\left(\frac{B(t+h) - B(t)}{h}\right)\psi.$$

By Theorem 7.2 in Part II, we have the following relations

$$\text{(9.4)} \quad \dot{B}(t)\psi = (\partial_t^* + \partial_t)\psi = x(t) \cdot \psi.$$

Remark 9.1. For each fixed $t \in T$, $x(t)\cdot$ does not operate on (L^2) naturally. However, if $\psi \in (L^2)$ is measurable with respect to the σ-field $\mathcal{B}_t$ generated by $\{B(s); s < t\}$, then the limit (9.3) exists in $\mathcal{H}^{(-1)}$ and coincides with $\partial_t^* \psi$ (see [13]).

Hida has introduced the renormalization of the polynomials of $\dot{B}(t)$'s as follows. Let $H_n(u; \alpha)$ be the Hermite polynomials given by (5.9). Let $t_1, \cdots, t_k$ be different points in $T = R$. Then the renormalization of $\dot{B}(t_1)^{n_1} \cdots \dot{B}(t_k)^{n_k}$ is given by

$$\text{(9.5)} \quad \mathcal{S}^{-1} \lim_{\Delta_j \downarrow \{t_j\}} \mathcal{S} \prod_{j=1}^{k} H_{n_j}\left(\frac{\Delta_j B}{|\Delta_j|}; \frac{1}{|\Delta_j|}\right)$$
$$= \mathcal{S}^{-1} \lim \prod_{j=1}^{k} \left(\frac{1}{|\Delta_j|} \int_{\Delta_j} \xi(u)\,du\right)^{n_j}$$
$$= \mathcal{S}^{-1} \prod_{j=1}^{k} \xi(t_j)^{n_j},$$

which is denoted by $\prod_{j=1}^{k} H_{n_j}(\dot{B}(t_j); 1/dt_j)$ formally. In our formulation, the convergence can be regarded as the strong convergence in $\mathcal{H}^*$ (or more precisely in $\mathcal{H}^{(-1)}$). By Theorem 6.1 and Lemma 7.3, (9.6) $$\prod_{j=1}^{k} H_{n_j}\Big(\dot{B}(t_j)\,;\frac{1}{dt_j}\Big)=\prod_{j=1}^{k}\partial_{t_j}^{*n_{j_1}}=:\prod_{j=1}^{k}(x(t_j)\cdot)^{n_j}:1.$$

Hida's generalized random measure $M_n(dt)$ can be expressed in the form;

$$\frac{1}{n_1!\cdots n_k!}\int dt_1\cdots dt_k f_n(t_1,\cdots,t_k):x(t_1)\cdot^{n_1}\cdots x(t_k)\cdot^{n_k}:1.$$

Partial derivatives of $\psi(x)=I_n(f_n)$ with $f_n\in\mathcal{E}^{\hat{\otimes}n}$ by $\dot{B}(t)$ have been defined by Hida [1] as follows

(9.7) $$\frac{\partial}{\partial\dot{B}(t)}\psi=\mathcal{T}^{-1}\Big(\Big\{\frac{\delta}{i\delta\xi(t)}\Big(\mathcal{T}\psi\cdot\exp\Big[\frac{1}{2}\|\xi\|_0^2\Big]\Big)\Big\}\exp\Big[-\frac{1}{2}\|\xi\|_0^2\Big]\Big).$$

By the relation (5.5), we can rewrite it as

(9.8) $$\frac{\partial}{\partial\dot{B}(t)}\psi=\mathcal{S}^{-1}\Big(\frac{\delta}{\delta\xi(t)}\Big)\mathcal{S}\psi,$$

here we note that this definition is identical with ∂_t in (6.1). Since we have formally that

$$\frac{\delta}{\delta\xi(t)}(\mathcal{S}\psi)(\xi)=\frac{\delta}{\delta\xi(t)}\int\psi(x+\xi)d\mu=\int\frac{\delta}{\delta\xi(t)}\psi(x+\xi)d\mu$$
$$=\int\frac{\partial}{\partial x(t)}\psi(x+\xi)d\mu=\mathcal{S}\Big(\frac{\partial}{\partial x(t)}\psi(x)\Big),$$

the relation (9.8) clarifies the reason why (9.7) gives us a natural differentiation. Actually, we have shown that ∂_t is a derivation (see Theorem 7.6).

§10. A simple application. First, we remark that the following theorem tells us a relation between Ito integral and our calculus.

Theorem 10.1. *Let $f(t,x)$ be $\mathcal{B}_t$-adapted function with $\int\|f(t,x)\|_0^2dt<\infty$, then*

(10.1) $$\int dt\partial_t^* f(t,x)=\int f(t,x)dB(t)$$

holds, where the right hand side means Ito's stochastic integral.

A. Shimizu [14] has discussed on the following bilinear stochastic differential equation by using Wiener expansion of its solution:

(10.2) $$\begin{cases} d\psi=\{\mathcal{L}_v\psi+a(t)\psi+b(t)\}dt+\{c(t)\psi+d(t)\}dB(t)\\ \psi(0,v,\cdot)=g(v),\ v\in R^d.\end{cases}$$

Suppose that the coefficients a, b, c and d are all continuous in t and that the operator $\mathcal{L}_v$ has a solution of the Cauchy problem:

(10.3) $$\frac{d}{dt}p(t,v)=\mathcal{L}_v p(t,v),\ p(0,v)=g(v).$$

We consider a solution ψ of (10.2) such that d/dt and $\mathcal{L}_v$ operate to ψ continuously in $\mathcal{H}^*$. Put $U(t,v\,;\xi)=(\mathcal{S}\psi(t,v,\cdot))(\xi)$. Since ∂_t^* corresponds to Ito integral, we have the following equation;

$$(10.4)\qquad \begin{cases} \dfrac{d}{dt}U=\{\mathcal{L}_v+a(t)+c(t)\xi(t)\}U+b(t)+d(t)\xi(t) \\ U(0,v;\xi)=g(v). \end{cases}$$

A solution of (10.4) can be expressed in the form

$$(10.5)\qquad U=p(t,v)V(t;\xi)+V(t,\xi)\int_0^t(b(s)+d(s)\xi(s))V(s,\xi)^{-1}ds,$$

where $V(t;\xi)=\exp\left[\int_0^t(a(s)+c(s)\xi(s))ds\right]$.

We have now to get the inverse formula of U under $\mathcal{S}$. Then by Lemma 5.9, we have

$$(10.6)\qquad \beta_t=\mathcal{S}^{-1}V=\exp\left[\int_0^t a(s)ds+\int_0^t c(s)dB(s)-\frac{1}{2}\int_0^t c(s)^2ds\right]$$

and together with Theorem 6.1, we have

$$\mathcal{S}^{-1}\left(\xi(s)\exp\left[\int_s^t c(r)\xi(r)dr\right]\right)=\partial_s^*\exp\left[\int_s^t c(r)dB(r)-\frac{1}{2}\int_s^t c(r)^2dr\right].$$

Since Theorems 7.6 and 6.1 can be extended to a more general case, it holds that

$$\partial_s^*(\beta_t\cdot\beta_s^{-1})=\beta_t(\partial_s^*\beta_s^{-1})-(\partial_s\beta_t)\cdot\beta_s^{-1}$$

and

$$\partial_s\beta_t=c(s)I_{[0,t]}(s)\beta_t.$$

Therefore we have a solution of (10.2);

$$(10.7)\qquad \psi=\beta_t\left\{p(t,v)+\int_0^t\beta_s^{-1}(b(s)-c(s)d(s))ds+\int_0^t\beta_s^{-1}d(s)dB(s)\right\}$$

which is given in [14]. The solution is unique if and only if the solution of (10.3), is unique.

§ 11. Formulae related to causality. An inverse formula of the transform $\mathcal{S}$ is given as follows from Theorems 6.5 and 8.2;

Theorem 11.1. *For a given* $U(\xi)\in\mathcal{F}$,

$$(11.1)\qquad \mathcal{S}^{-1}U=\sum_{k=0}^{\infty}\frac{1}{k!}\int U^{(k)}(0;t_1,\cdots,t_k)dW(t_1)\cdots dW(t_k).$$

Moreover, if $U(\xi)$ *is in* $\mathcal{F}^{(0)}$, *then* $U(\xi)$ *is* E_0*-differentiable arbitrary times and* $U^{(k)}(0;\eta_1,\cdots,\eta_k)$ *has an* L^2*-kernel* $U^{(k)}(0;t_1,\cdots,t_k)$ *which gives the formula* (11.1) *also.*

This is originally given by Hida-Ikeda [5]. Now we discuss the same case as in § 10. Let $\mathcal{B}_t$ be the σ-field generated by $\{B(s);s<t\}$ and let P_t be the orthogonal projection from $(L^2)=L^2(\mathcal{E}^*,\mu)$ to $(L_t^2)=\{\psi\in(L^2);\ \psi$ is $\mathcal{B}_t$-measurable$\}$.

Theorem 11.2. (i) *A given* ψ *in* $\mathcal{H}$ *is in* (L_t^2) *if and only if* $\partial_s\psi=0$ *for* $s>t$.

(ii) $$P_t\psi=\int_0^t\partial_s^*P_s\partial_s\psi ds+\int\psi d\mu$$

$$\psi=\int\partial_s^*P_s\partial_s\psi ds+\int\psi d\mu.$$

(iii) $\mathcal{S}P_t\psi(\xi)=\mathcal{S}\psi(I_{(-\infty,t)}\xi)$.

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