9. *Simple Singularities and Infinitesimally Symmetric Spaces*

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§0. **Introduction.** Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. We complexify $\mathfrak{g}_0, \mathfrak{k}_0$ and $\mathfrak{p}_0$ and denote them by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Let G be the adjoint group of $\mathfrak{g}$ and let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. Since $[\mathfrak{k}, \mathfrak{p}]$ is contained in $\mathfrak{p}, K$ naturally acts on $\mathfrak{p}$ and it follows from the theorem of Chevalley (see Helgason [6]) that the quotient space $\mathfrak{p}/K$ of $\mathfrak{p}$ by the action of K is isomorphic to an affine space C^n with a certain integer n. The nilpotent subvariety $N(\mathfrak{p})$ of $\mathfrak{p}$ is the totality of nilpotent elements of $\mathfrak{g}$ contained in $\mathfrak{p}$ and is also the fibre of $\pi(0)$, where $\pi:\mathfrak{p}\to\mathfrak{p}/K$ is the natural quotient map (see Kostant-Rallis [9]). For any element X of $N(\mathfrak{p})$ there is a linear subspace U_X of $\mathfrak{p}$ such that $S_X=X+U_X$ is transversal to the K-orbit of X at X. Then there appear singularities in the intersection of S_X with $N(\mathfrak{p})$.

The most typical example of the singularities appeared in this manner is a rational double point (or it is also called a two-dimensional simple singularity). We now explain this shortly. In this case, we take $\mathfrak{g}_0$ as a complex simple Lie algebra as a real one. Then $\mathfrak{k}_0$ is a compact real form of $\mathfrak{g}_0$ and $\mathfrak{k}$ is isomorphic to $\mathfrak{p}$ and the action of K on $\mathfrak{p}\cong\mathfrak{k}$ is nothing but the adjoint action. Under the situation the results of Brieskorn [4] and Slodowy [10] assure that if we take X as a subregular nilpotent element of $N(\mathfrak{p})$ then the variety $S_X\cap N(\mathfrak{p})$ becomes a surface and the singularity of the surface is a rational double point. In particular, if the root system of $\mathfrak{k}$ is homogeneous, that is, the type of $\mathfrak{k}$ is one of A_l, D_l, E_6, E_7 or E_8, then the singularity of $S_X\cap N(\mathfrak{p})$ is a rational double point of the corresponding type:

$$(A_l)\quad x^{l+1}+y^2+z^2=0 \qquad (l\geqq 1),$$
$$(D_l)\quad x^{l-1}+xy^2+z^2=0 \qquad (l\geqq 4),$$
$$(E_6)\quad x^4+y^3+z^2=0,$$
$$(E_7)\quad x^3y+y^3+z^2=0,$$
$$(E_8)\quad x^5+y^3+z^2=0,$$

and moreover the restriction $\delta: S_X\to\mathfrak{p}/K$ of π to S_X is a semiuniversal deformation of the rational double point.

In the present note, we treat the case when $\mathfrak{g}_0$ is a normal real form of a complex simple Lie algebra and examine the singularity of

$S_X \cap N(\mathfrak{p})$ corresponding to a "subregular" nilpotent element X of $N(\mathfrak{p})$. The main results are stated in Theorems 3 and 4, which explain a connection between the symmetries of a rational double point and Cartan involutions.

§ 1. Subregular nilpotent elements of an infinitesimally symmetric space. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{g}_0$ a real form of $\mathfrak{g}$ and θ a Cartan involution of $\mathfrak{g}_0$ (see Helgason [6]). We extend θ to $\mathfrak{g}$ as a complex linear automorphism. Set $\mathfrak{k}=\{X \in \mathfrak{g}\,;\,\theta(X)=X\}$ and $\mathfrak{p}=\{X \in \mathfrak{g}\,;\,\theta(X)=-X\}$. Then we obtain the complexified Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. We call $\mathfrak{p}$ an infinitesimally symmetric space in this note. For later convenience, we say that $\mathfrak{p}$ is of the normal type if the corresponding $\mathfrak{g}_0$ is a normal real form of $\mathfrak{g}$ (see Helgason [6]). Let G be the adjoint group of $\mathfrak{g}$ and let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. For any element k of K, $\mathrm{Ad}\,(k)|_{\mathfrak{p}}$ is an automorphism of $\mathfrak{p}$ and in this way K acts on $\mathfrak{p}$.

The G-orbits of nilpotent elements of $\mathfrak{g}$ are already classified by Dynkin (see Bala-Carter [3] and Steinberg [11]). Among all the nilpotent orbits, we pay attention to two kinds of orbits. A nilpotent element X is called regular (or subregular) if $\dim Z_{\mathfrak{g}}(X)=l$ (or $\dim Z_{\mathfrak{g}}(X)=l+2$), where $Z_{\mathfrak{g}}(X)$ is the centralizer of X in $\mathfrak{g}$ and l is the rank of $\mathfrak{g}$. Then we infer that the set of regular (or subregular) nilpotent elements of $\mathfrak{g}$ is a single G-orbit and we denote it by $N_{r.}(\mathfrak{g})$ (or $N_{s.r.}(\mathfrak{g})$). An element of $\mathfrak{p}$ is called nilpotent if it is nilpotent as an element of $\mathfrak{g}$. We now discuss the K-orbit structure of nilpotent elements of $\mathfrak{p}$. Vinberg [12] already classified the K-orbits of them (see also Kostant-Rallis [9]). But when $\mathfrak{p}$ is of the normal type, we obtain more detailed information on K-orbits of arbitrary elements in $\mathfrak{p}$.

Theorem 1. *Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a complexified Cartan decomposition. Assume that $\mathfrak{p}$ is of the normal type. Then for any G-orbit $\mathcal{O}$ of $\mathfrak{g}$, we have $\mathcal{O}\cap\mathfrak{p}\neq\phi$.*

Corollary. *If $\mathfrak{p}$ is of the normal type, then we have $N_{r.}(\mathfrak{g})\cap\mathfrak{p}\neq\phi$ and $N_{s.r.}(\mathfrak{g})\cap\mathfrak{p}\neq\phi$.*

Remark. (1) The claim of the theorem does not hold in general. In particular, there is an infinitesimally symmetric space $\mathfrak{p}$ which is not of the normal type and such that $N_{r}(\mathfrak{g})\cap\mathfrak{p}=\phi$ and $N_{s.r.}(\mathfrak{g})\cap\mathfrak{p}=\phi$.

(2) In contrast to G-orbits of $N_{r.}(\mathfrak{g})$ and $N_{s.r.}(\mathfrak{g})$, $N_{r.}(\mathfrak{g})\cap\mathfrak{p}$ is not a single K-orbit and so is $N_{s.r.}(\mathfrak{g})\cap\mathfrak{p}$.

In the sequel we always assume that $\mathfrak{p}$ is of the normal type. Under the assumption, the following holds.

Lemma 1. (1) *For any element X of $N_{r.}(\mathfrak{g})\cap\mathfrak{p}$, we have*

$$\dim Z_{\mathfrak{g}}(X)\cap\mathfrak{p}=l \quad \textit{and} \quad \dim Z_{\mathfrak{g}}(X)\cap\mathfrak{k}=0.$$

(2) *For any element X of $N_{s.r.}(\mathfrak{g})\cap\mathfrak{p}$, we have* $\dim Z_{\mathfrak{g}}(X)\cap\mathfrak{p}=l+1$ *and* $\dim Z_{\mathfrak{g}}(X)\cap\mathfrak{k}=1$.

Let X be any element of $\mathfrak{g}$. We denote by $G\cdot X$ the G-orbit through X. Let $T_X(G\cdot X)$ be the tangent space to $G\cdot X$ at X. We identify $T_X(G\cdot X)$ with a subspace of $\mathfrak{g}$ and take a linear complement U_X of $T_X(G\cdot X)$ in $\mathfrak{g}$, that is, $\mathfrak{g}=T_X(G\cdot X)\oplus U_X$. The affine subspace $S_X=X+U_X$ is transversal to $G\cdot X$ at X in $\mathfrak{g}$ and by this reason we call S_X a transversal slice of $G\cdot X$ at X. In this terminology we have the following lemma which is essential to proving the main theorems.

Lemma 2. *Let X be an element of $N_{s.r.}(\mathfrak{g})\cap\mathfrak{p}$. Then there exists a $(-\theta)$-stable transversal slice $S_X=X+U_X$ of $G\cdot X$ at X in $\mathfrak{g}$ such that* $\dim(U_X\cap\mathfrak{k})=\mathrm{codim}_{S_X}(S_X\cap\mathfrak{p})=1$ *and that $S_X\cap\mathfrak{p}$ is also transversal to the K-orbit of X at X in $\mathfrak{p}$.*

Due to this lemma we choose a coordinate system $(\lambda_1,\cdots,\lambda_{l+1},\mu)$ of S_X such that the restriction $(-\theta)|_{S_X}$ acts on S_X in the following manner: $(\lambda_1,\cdots,\lambda_{l+1},\mu)\mapsto(\lambda_1,\cdots,\lambda_{l+1},-\mu)$. We call this system a good coordinate system.

§2. Subregular nilpotent elements and simple singularities. In this section situations and notations are the same as above.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For convenience we assume that $\mathfrak{h}$ is contained in $\mathfrak{p}$. This is actually possible, because $\mathfrak{p}$ is of the normal type (see Helgason [6]). Let γ be the adjoint quotient of $\mathfrak{g}$ onto $\mathfrak{h}/W$ (see Slodowy [10, p. 37]). We now mention a result of Brieskorn-Slodowy. Let X be an element of $N_{s.r.}(\mathfrak{g})$ and let S_X be a transversal slice of $G\cdot X$ at X in $\mathfrak{g}$.

Theorem 2 (Brieskorn [4], Slodowy [10, p. 136]). *There is a polynomial $F_\xi(x,y,z)$ of three variables x,y,z and with parameters ξ in $\mathfrak{h}/W$ such that $\gamma^{-1}(\xi)\cap S_X$ is locally biholomorphic to $\{(x,y,z)\in\boldsymbol{C}^3;\ F_\xi(x,y,z)=0\}$ for any element ξ of $\mathfrak{h}/W$. Corresponding to the type of $\mathfrak{g}$, $F_\xi(x,y,z)$ is given by the following table:*

A_l: $x^{l+1}+y^2+z^2+\xi_2x^{l-1}+\xi_3x^{l-2}+\cdots+\xi_lx+\xi_{l+1}=0$ $(l\geqq1)$,

B_l: $x^{2l}+y^2+z^2+\xi_2x^{2l-2}+\xi_4x^{2l-4}+\cdots+\xi_{2l-2}x^2+\xi_{2l}=0$ $(l\geqq2)$,

C_l: $x^l+xy^2+z^2+\xi_2x^{l-1}+\xi_4x^{l-2}+\cdots+\xi_{2l-2}x+\xi_{2l}=0$ $(l\geqq3)$,

D_l: $x^{l-1}+xy^2+z^2+\xi_2x^{l-2}+\xi_4x^{l-3}+\cdots+\xi_{2l-4}x+\xi_{2l-2}+\xi'_ly=0$ $(l\geqq4)$,

E_6: $x^4+y^3+z^2+\xi_2x^2y+\xi_5xy+\xi_6x^2+\xi_8y+\xi_9x+\xi_{12}=0$,

E_7: $x^3y+y^3+z^2+\xi_2x^4+\xi_6x^3+\xi_8xy+\xi_{10}x^2+\xi_{12}y+\xi_{14}x+\xi_{18}=0$,

E_8: $x^5+y^3+z^2+\xi_2x^3y+\xi_8x^2y+\xi_{12}x^3+\xi_{14}xy+\xi_{18}x^2+\xi_{20}y+\xi_{24}x+\xi_{30}=0$,

F_4: $x^4+y^3+z^2+\xi_2x^2y+\xi_6x^2+\xi_8y+\xi_{12}=0$,

G_2: $x^3+y^3+z^2+\xi_2xy+\xi_6=0$.

(In the table the indices of the parameters $\xi_2,\cdots$ (and ξ'_l in the case D_l) denote the weights of them.)

The surface singularity $F_0(x,y,z)=0$ is called a rational double point or a (two-dimensional) simple singularity by Arnol'd (see [1], [2]) *and its deformation family $\{F_\xi(x,y,z)=0;\ \xi\in\mathfrak{h}/W\}$ is a semiuniversal* *deformation of* $F_0(x, y, z)=0$.

Now we observe the fact that $F_\xi(x, y, z)=0$ has the following symmetries: $z \mapsto -z$ (in all the cases), $x \mapsto -x$ (in the cases B_l, F_4), $y \mapsto -y$ (in the case C_l), $x \leftrightarrow y$ (in the case G_2). We give an interpretation of these symmetries in the following main theorems.

Theorem 3. *There is a subregular nilpotent element* X *of* $N_{s.r.}(\mathfrak{g}) \cap \mathfrak{p}$ *such that if we choose a transversal slice* S_X *as in Lemma* 2 *and a good coordinate system* $(\lambda_1, \cdots, \lambda_{l+1}, \mu)$ *of* S_X, *the involution*

$$(-\theta)|_{S_X}: (\lambda_1, \cdots, \lambda_{l+1}, \mu) \mapsto (\lambda_1, \cdots, \lambda_{l+1}, -\mu)$$

induces the symmetry $z \mapsto -z$ *on the surface* $\gamma^{-1}(\xi) \cap S_X$ *for any element* ξ *of* $\mathfrak{h}/W$.

Theorem 4. *Assume that the root system of* $\mathfrak{g}$ *is inhomogeneous, that is, the type of* $\mathfrak{g}$ *is one of* B_l, C_l, F_4 *and* G_2. *Then there is another subregular nilpotent element* Y *of* $N_{s.r.}(\mathfrak{g}) \cap \mathfrak{p}$ *not* K*-conjugate to* X *in Theorem* 3 *such that if we choose a transversal slice* S_Y *as in Lemma* 2 *and a good coordinate system* $(\lambda_1, \cdots, \lambda_{l+1}, \mu)$, *the involution*

$$(-\theta)|_{S_Y}: (\lambda_1, \cdots, \lambda_{l+1}, \mu) \mapsto (\lambda_1, \cdots, \lambda_{l+1}, -\mu)$$

induces the symmetry $x \mapsto -x$ (*in the cases* B_l, F_4), $y \mapsto -y$ (*in the case* C_l), $x \leftrightarrow y$ (*in the case* G_2) *on the surface* $\gamma^{-1}(\xi) \cap S_Y$ *for any element* ξ *of* $\mathfrak{h}/W$.

The set of the fixed points of $\gamma^{-1}(\xi) \cap S_X$ (or $\gamma^{-1}(\xi) \cap S_Y$) by the involution $(-\theta)|_{S_X}$ (or $(-\theta)|_{S_Y}$) is a deformation of a one-dimensional simple singularity in the sense of Arnol'd (see [1]). In particular we obtain the following

Corollary (to Theorem 3). *If the root system of* $\mathfrak{g}$ *is homogeneous, that is, the type of* $\mathfrak{g}$ *is one of* A_l, D_l, E_6, E_7 *and* E_8, *then the restriction* $\delta: S_X \cap \mathfrak{p} \to \mathfrak{h}/W$ *of* γ *to the intersection* $S_X \cap \mathfrak{p}$ *is a semiuniversal deformation of the one-dimensional simple singularity of the corresponding type*:

(A_l) $x^{l+1}+y^2=0$ $(l \geqq 1)$,
(D_l) $x^{l-1}+xy^2=0$ $(l \geqq 4)$,
(E_6) $x^4+y^3=0$,
(E_7) $x^3y+y^3=0$,
(E_8) $x^5+y^3=0$.

The proofs of Theorems 3 and 4 need the results of Slodowy [10], Bala-Carter [3] and Elkington [5].

An extended version of this note and detailed proofs will be published elsewhere.

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