86. *On Richardson Classes of Unipotent Elements in Semisimple Algebraic Groups*

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Let G be a connected semisimple algebraic group over an algebraically closed field K of characteristic zero. Let P be a parabolic subgroup of G and U_P its unipotent radical. A unipotent class in G is called a Richardson class corresponding to P if it intersects U_P densely [6]. We study here the correspondence between Richardson classes and parabolic subgroups in detail. Note here that, as is shown in [3], we naturally encounter the notion of Richardson classes or more generally that of induced classes, in the study of Fourier transform of unipotent orbital integrals on a connected semisimple Lie group.

§1. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Σ the root system of $(\mathfrak{g}, \mathfrak{h})$, Σ^+ the set of all positive roots in Σ, and Π the set of all simple roots. For a subset Γ of Π, we define a standard parabolic subgroup $P(\Gamma)$ of G as follows. Put $\Gamma^\perp=\{X \in h\,;\, \gamma(X)=0\ (\gamma \in \Gamma)\}$, and $L(\Gamma)=\{g \in G\,;\, \mathrm{Ad}(g)X=X\ (X \in \Gamma^\perp)\}$. Further let $\langle\Gamma\rangle$ be the set of all roots in Σ expressed as integral linear combinations of $\gamma \in \Gamma$, U_α the one-dimensional unipotent subgroup corresponding to $\alpha \in \Sigma$, and $U(\Gamma)$ the subgroup generated by U_α's for $\alpha \in \Sigma^+ - \langle\Gamma\rangle$. Then $P(\Gamma)=L(\Gamma)U(\Gamma)$ is a parabolic subgroup of G with a Levi subgroup $L(\Gamma)$ and the unipotent radical $U(\Gamma)$. Note that $\langle\Gamma\rangle$ is the root system of $L(\Gamma)$ and $\Sigma^+ - \langle\Gamma\rangle$ is the ideal $I(\Pi-\Gamma)$ of Σ^+ generated by $\Pi-\Gamma$ in the sense of [7, §2].

For $\Gamma, \Gamma' \subset \Pi$, we define "$\Gamma \sim \Gamma'$ *in* Σ" if the Richardson classes corresponding to $P(\Gamma)$ and $P(\Gamma')$ coincide with each other. Remark that $\Gamma \sim \Gamma'$ here is equivalent to $I(\Pi-\Gamma) \sim I(\Pi-\Gamma')$ in [7, §2].

Rewriting Theorem 1.7 in [5], we have the following.

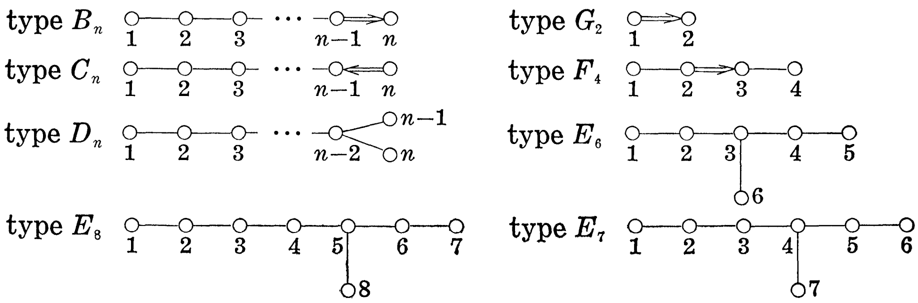
Theorem 1 [5]. *Let W be the Weyl group of the root system Σ, and let $\Gamma, \Gamma' \subset \Pi$. If $w\Gamma=\Gamma'$ for some $w \in W$, then $\Gamma \sim \Gamma'$.*

We also have the following general theorem.

Theorem 2. *Let Π_1, Π_2 be two subsets of Π orthogonal to each other, and let Γ_i, Γ'_i be two subsets of Π_i for $i=1, 2$. Assume that $\Gamma_i \sim \Gamma'_i$ in the root system $\langle\Pi_i\rangle$ for $i=1, 2$. Then $\Gamma=\Gamma_1 \cup \Gamma_2 \sim \Gamma'=\Gamma'_1 \cup \Gamma'_2$ in $\Sigma=\langle\Pi\rangle$. Here $\Pi_2=\phi$ may be admitted.*

§2. We call a subsystem of the relations $\Gamma \sim \Gamma'$ in various (Π, Σ)

fundamental if it generates under the properties of $\sim$ in Theorems 1 and 2 the whole relation $\sim$. We list up here a system of fundamental relations. The simple roots α_i's in each type of irreducible root systems are numbered as follows.



We denote simply by Z a root system of type Z.

(1) In B_n or C_n with $n=3k-1$, $k\geqslant 1$, $(\Pi-\{\alpha_{2k-1}\})\sim(\Pi-\{\alpha_{2k}\})$.

(2) In D_4, $\{\alpha_1,\alpha_2\}\sim\{\alpha_1,\alpha_3,\alpha_4\}$.

(3) In D_n with $n=3k+1$, $k\geqslant 2$, $(\Pi-\{\alpha_{2k}\})\sim(\Pi-\{\alpha_{2k+1}\})$.

(4) In E_6, $(\Pi-\{\alpha_3\})\sim(\Pi-\{\alpha_4,\alpha_6\})$.

(5) In E_8, $(\Pi-\{\alpha_4\})\sim(\Pi-\{\alpha_6,\alpha_8\})$.

(6) In F_4, $\{\alpha_1,\alpha_2,\alpha_4\}\sim\{\alpha_1,\alpha_3,\alpha_4\}\sim\{\alpha_2,\alpha_3\}$.

(7) In G_2, $\{\alpha_1\}\sim\{\alpha_2\}$.

Let σ be the unique non-trivial automorphism of the Dynkin diagram of D_{2k+1} or of E_6.

(8) In D_{2k+1} $(k\geqslant 2)$ or in E_6, $\sigma\Gamma\sim\Gamma$ for any $\Gamma\subset\Pi$.

One of our main results is the following.

Theorem 3. *Under the properties of the relation $\sim$ in Theorems 1 and 2, the system of relations* (1)–(8) *is fundamental.*

To get a smaller system consisting of mutually independent relations, (8) is replaced by (8′) below. Further the set of relations in Theorem 1 can be replaced by (9) below. That is,

Theorem 4. *The system of relations* (1)–(7), (8′) *and* (9) *generates the whole relation $\sim$ under the property of $\sim$ in Theorem* 2.

(8′) In $D_{2k+1}(k\geqslant 2)$, $(\Pi-\{\alpha_{2k}\})\sim(\Pi-\{\alpha_{2k+1}\})$. In E_6, $(\Pi-\{\alpha_1\})\sim(\Pi-\{\alpha_5\})$, $(\Pi-\{\alpha_2\})\sim(\Pi-\{\alpha_4\})$.

(9) In A_n, if $w\Gamma=\Gamma'$ for $\Gamma,\Gamma'\subset\Pi$ with some $w\in W$, the Weyl group of A_n, then $\Gamma\sim\Gamma'$.

§ 3. Let us now determine the Richardson class corresponding to $P(\Gamma)$ by means of Γ. To do so, we turn to $\mathfrak{g}$ instead of G for convenience of statement. After Hotta-Springer [4, p. 119], we call a nilpotent element $X\in\mathfrak{g}$ *of parabolic type* with respect to $P(\Gamma)$ if its class $G(X)=\{\mathrm{Ad}(g)X\,;\,g\in G\}$ intersects $\mathfrak{u}(\Gamma)$ densely, where $\mathfrak{u}(\Gamma)$

denotes the sum of the root subspaces $\mathfrak{g}_\alpha$ over $\alpha \in \Sigma^+ - \langle \Gamma \rangle$. For classical types B_n, C_n, and D_n, it is convenient to use as a parameter for the class $G(X)$ the Jordan normal form of X under conjugations of $GL(N, K)$ with $N=2n+1$, $2n$ or $2n$ respectively. For exceptional types, the (weighted) Dynkin diagram of X [8, p. 243] is used as a parameter.

Classical types. Let X be conjugate under $GL(N, K)$ to $J(p_1) \oplus J(p_2) \oplus \cdots \oplus J(p_s)$, $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_s$, $p_1+p_2+\cdots+p_s=N$, where $J(p)$ is the $p \times p$ Jordan matrix with entries 1 just above the diagonal and zero except there (cf. [3, § 2]). Then we say that X is *of Jordan type* $(p_1, p_2, \cdots, p_s)$. Note that in D_n with n even, if all p_i are even, exactly two classes have the same Jordan type $(p_1, p_2, \cdots, p_s)$, and that the Dynkin diagram of X is obtained from $(p_1, p_2, \cdots, p_s)$ as in [8, pp. 263–264].

One knows that for B_n or D_n,

(BD1) the multiplicity of any even integer in p_i's is even.

Theorem 5. *A nilpotent element X in B_n or D_n of Jordan type $(p_1, p_2, \cdots, p_s)$ is of parabolic type if and only if it satisfies the condition* (BD2) *below.*

(BD2) *Let p_t be even and p_j are all odd for $j<t$. Then, for p_j with $j \geqslant t$,* (i) *the multiplicity of any odd integer in p_i's is at most two, and* (ii) *if $p_i, p_{i+1}, \cdots, p_{j-1}$ are all odd and of multiplicity* 1, *and $i=1$* (resp. *$j-1=s$*) *or p_{i-1}* (resp. *p_j*) *is even or multiplicity* 2, *then $i-j$ is even.*

The condition (BD2) can not be expressed in a simple manner by means of the Dynkin diagram. The analogous result for type C_n was given in [3, § 5]. These are essentially contained in [2].

Let X be of parabolic type with respect to $P(\Gamma)$. Then, we can determine the Jordan type of X by means of Γ similarly as for type C_n in [3, § 5]. This is nothing but the Spaltenstein mapping in [2, p. 225]. Note that, for D_n with n even, an additional discussion is necessary when all p_i are even.

Exceptional types. A nilpotent element $X \in \mathfrak{g}$ is called *even* if the weights in its Dynkin diagram are all 0 or 2. Every even nilpotent element is of parabolic type and the corresponding Γ is $\Gamma_X = \{\alpha \in \Pi$; the weight of X for α is zero$\}$, or $\Gamma \sim \Gamma_X$. We give a complete list of non-even nilpotent classes of parabolic type and one of the corresponding subsets $\Gamma \subset \Pi$ for each class. We list up together with the Dynkin diagram of X the type of a minimal regular subalgebra containing X (cf. [1, pp. 176–185]), which is used in [7, Tables 1–3, pp. 446–449] as a name of the class $G(X)$.

	Dynkin diagram	Name of class	Corresponding Γ
type F_4	$1\ 0\ 1\ 2$	C_3	$\{\alpha_1, \alpha_2\}$
type E_6	$1\ 0\ \underset{0}{0}\ 0\ 1$	$2A_1$	$\Pi-\{\alpha_1\}$
	$0\ 1\ \underset{0}{0}\ 1\ 0$	A_2+2A_1	$\Pi-\{\alpha_2\}$
	$1\ 0\ \underset{2}{0}\ 0\ 1$	A_3	$\Pi-\{\alpha_1, \alpha_2\}$
	$1\ 1\ \underset{1}{0}\ 1\ 1$	A_4+A_1	$\Pi-\{\alpha_2, \alpha_4\}$
	$1\ 1\ \underset{2}{0}\ 1\ 1$	$D_5(a_1)$	$\{\alpha_1, \alpha_2, \alpha_4\}$
type E_7	$1\ 0\ 0\ \underset{1}{0}\ 1\ 0$	$D_4(a_1)+A_1$	$\Pi-\{\alpha_6, \alpha_7\}$
	$0\ 1\ 0\ \underset{0}{1}\ 0\ 1$	A_4+A_1	$\Pi-\{\alpha_5, \alpha_7\}$
	$0\ 1\ 0\ \underset{0}{1}\ 0\ 2$	$D_5(a_1)$	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
	$0\ 1\ 1\ \underset{1}{0}\ 1\ 2$	D_5+A_1	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$
	$2\ 0\ 1\ \underset{1}{0}\ 1\ 2$	$D_6(a_1)$	$\{\alpha_1, \alpha_2, \alpha_3\}$
type E_8	$0\ 0\ 1\ 0\ \underset{0}{0}\ 1\ 0$	$A_4+A_2+A_1$	$\Pi-\{\alpha_6\}$
	$0\ 0\ 1\ 0\ \underset{0}{1}\ 0\ 1$	A_6+A_1	$\Pi-\{\alpha_5\}$
	$2\ 1\ 0\ 0\ \underset{1}{0}\ 1\ 0$	$D_6(a_1)$	$\Pi-\{\alpha_6, \alpha_7, \alpha_8\}$
	$1\ 0\ 1\ 0\ \underset{0}{1}\ 0\ 1$	$D_7(a_2)$	$\Pi-\{\alpha_5, \alpha_7\}$
	$2\ 0\ 1\ 0\ \underset{0}{1}\ 0\ 1$	$E_6(a_1)+A_1$	$\Pi-\{\alpha_5, \alpha_7, \alpha_8\}$
	$2\ 0\ 1\ 0\ \underset{0}{1}\ 0\ 2$	D_6+A_1	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
	$2\ 2\ 0\ 1\ \underset{1}{0}\ 1\ 2$	$E_7(a_1)$	$\{\alpha_1, \alpha_2, \alpha_3\}$

§4. **Sketch of the proof.** (1°) The property (8) in §2 follows from the invariance of the Dynkin diagram of any nilpotent element under σ (see [8, p. 264] for D_{2n+1}, and [2, Table 18, p. 178] for E_6).

(2°) For classical types we first prove the results in §3 in a similar way as in [3, §5]. (Note that the uniqueness of maximal elements modulo conjugacy in Lemma 5.5 in [3] follows from that of the class which intersects $\mathfrak{u}(\Gamma)$ densely.) We can also get this result from that in [2]. Now, Theorem 4 for this case is proved by the process itself of determining explicitly the Jordan type of X of parabolic type with respect to $P(\Gamma)$ by means of Γ.

(3°) For exceptional types, we discuss type by type inductively according to the ranks of root systems. We first classify the subsets Γ of Π under (1)–(3), (8′), (9) and Theorem 2. Then for every representative Γ, we determine the diagram of an X corresponding to $P(\Gamma)$, using the following.

(i) The dimension of $G(X)$ is equal to $2\dim\mathfrak{u}(\Gamma)$ by Theorem

1.3(a) in [6]. The dimension of any class is given by the Dynkin diagram of its elements as $\dim \mathfrak{g} - \dim \mathfrak{g}(0) - \dim \mathfrak{g}(1)$, where $\mathfrak{g}(j)$ is the space spanned by $\mathfrak{g}_\alpha$'s for roots α with weight j [8, p. 241].

(ii) When there exist some number (2 or 3) of classes of dimension $2 \dim \mathfrak{u}(\Gamma)$, we find by calculations, for one of them C, its representative $X \in \mathfrak{g}(2)$ of the form $X = \Sigma_{\alpha \in \Psi} X_\alpha$ with $X_\alpha \in \mathfrak{g}_\alpha$, $\neq 0$, and $\Psi \subset \Sigma(\mathfrak{g}(2))$ consisting of linearly independent roots (cf. [8, p. 246]), and a $w \in W$ such that $w\Psi \subset \Sigma(\mathfrak{u}(\Gamma))$ for an appropriately chosen representative Γ under the above classification. Here for a subspace $\mathfrak{a}$ of $\mathfrak{g}$, we put $\Sigma(\mathfrak{a}) = \{\alpha \in \Sigma\,;\, \mathfrak{g}_\alpha \subset \mathfrak{a}\}$. This proves that C is just the class corresponding to $P(\Gamma)$.

The validity of Theorem 4 for this case can be seen now.

Remark. More elementarily, we need not use Theorem 1.3(a) in [6]. In fact, even when there exists only one class of dimension $2 \dim \mathfrak{u}(\Gamma)$, we can proceed as in (ii), and then $\dim G(X) = 2 \dim \mathfrak{u}(\Gamma)$ is obtained as its consequence.

We shall give two examples to explain the situation in (ii) more in detail.

Example 1. Type E_8. Take $\Gamma = \Pi - \{\alpha_5, \alpha_7\}$ in the list in § 3. To prove that the Richardson class corresponding to $P(\Gamma)$ is the class with Dynkin diagram $\overset{1010101}{0}$ and named $D_7(a_2)$ in [7, p. 448], we proceed as follows. (Other classes of the same dimension, 216, are with diagram $\overset{2100012}{1}$ and named D_6, and with $\overset{2220002}{0}$ named E_6.) Let $\Gamma' = \Pi - \{\alpha_4, \alpha_7\}$, then $\Gamma' \sim \Gamma$ in Σ by (3) for D_7. We can find a $\Psi \subset \Sigma(\mathfrak{g}(2)) \cap \Sigma(\mathfrak{u}(\Gamma'))$ such that $X = \Sigma_{\alpha \in \Psi} X_\alpha$ is a nilpotent element of type $D_5 + A_3$ and that $\mathrm{ad}(X)\mathfrak{g}(0) = \mathfrak{g}(2)$ (cf. [1, Table 20, p. 184]). The Ψ is given below, where a root is expressed by its coefficients with respect to simple roots.

$$\overset{0000111}{0},\ \overset{0011100}{0},\ \overset{0001111}{1},\ \overset{0001210}{1},\ \overset{0011110}{1},\ \overset{0111110}{0},\ \overset{0111100}{1},\ \overset{1111000}{0}.$$

Example 2. Type E_8. The most complicated was to establish the relation (5) in § 2. Put $\Gamma = \Pi - \{\alpha_4\}$, $\Gamma' = \Pi - \{\alpha_6, \alpha_8\}$, and let C and C' be the Richardson classes corresponding to $P(\Gamma)$ and $P(\Gamma')$ respectively. Then they have the same dimension 208, and the class C is even, with Dynkin diagram $\overset{0002000}{0}$ and named $2A_4$. There is another unipotent class of the same dimension, with Dynkin diagram $\overset{2101001}{0}$ and named $D_5 + A_1$.

To prove $C' = C$, first we study $\Sigma(\mathfrak{g}(2))$ for C. It contains five kinds of subset Ψ such that $X = \Sigma_{\alpha \in \Psi} X_\alpha$ satisfies $\mathrm{ad}(X)\mathfrak{g}(0) = \mathfrak{g}(2)$, and that the minimal regular subalgebra containing X is of type $2A_4$, $A_5 + A_2 + A_1$, $D_4(a_1) + D_4$, $D_5(a_1) + A_3$, or $D_6(a_2) + 2A_1$. (Here "of type $D_4(a_1) + D_4$" means that it is of type $D_4 + D_4$ as algebra and contains X as its semiregular element of type $D_4(a_1) + D_4$.) We give examples of Ψ one for each type.

Let β_i $(1\leqslant i\leqslant 6)$, γ_j $(1\leqslant j\leqslant 5)$, and δ_k $(1\leqslant k\leqslant 5)$ be positive roots given as follows:

$$\genfrac{}{}{0pt}{}{0001000}{0},\ \genfrac{}{}{0pt}{}{0011100}{0},\ \genfrac{}{}{0pt}{}{0001210}{1},\ \genfrac{}{}{0pt}{}{0011110}{1},\ \genfrac{}{}{0pt}{}{0111110}{0},\ \genfrac{}{}{0pt}{}{0111111}{1};$$

$$\genfrac{}{}{0pt}{}{1111100}{1},\ \genfrac{}{}{0pt}{}{1111111}{0},\ \genfrac{}{}{0pt}{}{0011221}{1},\ \genfrac{}{}{0pt}{}{0111100}{1},\ \genfrac{}{}{0pt}{}{0001211}{1};$$

$$\genfrac{}{}{0pt}{}{0001221}{1},\ \genfrac{}{}{0pt}{}{0111211}{1},\ \genfrac{}{}{0pt}{}{1111111}{1},\ \genfrac{}{}{0pt}{}{0011111}{0},\ \genfrac{}{}{0pt}{}{1111211}{1}.$$

Then Ψ, one for each type is given as

$2A_4$: $\beta_1, \beta_2, \beta_3, \beta_5, \beta_6, \gamma_1, \gamma_2, \gamma_3$;

$A_5+A_2+A_1$: $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \gamma_1, \gamma_2$;

$D_6(a_2)+2A_1$: $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \gamma_2, \gamma_4, \gamma_5$;

$D_5(a_1)+A_3$: $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \delta_1, \delta_2, \delta_3$;

$D_4(a_1)+D_4$: $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \delta_4, \delta_5$.

(To prove $\mathrm{ad}(X)\mathfrak{g}(0)=\mathfrak{g}(2)$ for type $D_5(a_1)+A_3$, we use Mizuno's result on $\mathrm{sgn}(N_{\alpha\beta})$ [7, Table 12, p. 460] in case of type E_7.)

On the other hand, we proved that there is no subset Ψ' of $\Sigma(\mathfrak{u}(\Gamma'))$ which is of type $2A_4$. However we can find many Ψ' of type $A_5+A_2+A_1$, one of which is given below:

$$\genfrac{}{}{0pt}{}{0000000}{1},\ \genfrac{}{}{0pt}{}{0000110}{0},\ \genfrac{}{}{0pt}{}{0001111}{0},\ \genfrac{}{}{0pt}{}{0011211}{1},\ \genfrac{}{}{0pt}{}{0112210}{1},\ \genfrac{}{}{0pt}{}{0122221}{1},\ \genfrac{}{}{0pt}{}{1122210}{1},\ \genfrac{}{}{0pt}{}{1222211}{1}.$$

This Ψ' is automatically conjugate under W to the Ψ of type $A_5+A_2+A_1$ above.

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