84. *On the Neighbourhood of a Hopf Surface*

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0. Introduction. Let S be a non-singular compact complex surface imbedded in a complex manifold of dimension 3. As a differentiable manifold, the structure of the tubular neighbourhood of S is determined by its normal bundle. But, in general, the complex analytic structure of the tubular neighbourhood of S cannot be determined by the normal bundle.

In this note we shall state theorems on the complex analytic structure of the tubular neighbourhood of a Hopf surface imbedded in a complex manifold of dimension 3. In this case pseudoconvexity of the domain of holomorphy and the Silov boundary of the domain in $\boldsymbol{C}^2$ play essential roles. Such circumstance cannot occur in case of the tubular neighbourhood of a compact complex curve imbedded in a complex surface.

1. Statement of results. Definition 1.1. A non-singular compact complex surface is called a Hopf surface, if its universal covering surface is biholomorphic to $\boldsymbol{C}^2-O$ (O is the origin of $\boldsymbol{C}^2$). If moreover the fundamental group of a Hopf surface is an infinite cyclic group, we call the surface a primary Hopf surface.

The following facts are well-known ([3]).

(a) Every primary Hopf surface has the following normal form:

$$S_{\alpha_1\alpha_2\lambda}=\boldsymbol{C}^2-O/\langle g\rangle, \qquad g(z_1, z_2)=(\alpha_1 z_1+\lambda z_2^m, \alpha_2 z_2),$$

where $\langle g\rangle$ denotes the group of automorphisms of $\boldsymbol{C}^2-O$ generated by g, (z_1, z_2) denote the standard coordinates of $\boldsymbol{C}^2$ and $\alpha_i\in\boldsymbol{C}^*$ $(i=1,2)$, $\lambda\in\boldsymbol{C}$, $m\in\boldsymbol{Z}^+$ satisfying $0<|\alpha_1|\leqq|\alpha_2|<1$, $(\alpha_1-\alpha_2^m)=0$. If $\lambda_1, \lambda_2\neq 0$, then $S_{\alpha_1\alpha_2\lambda_1}$ and $S_{\alpha_1\alpha_2\lambda_2}$ are biholomorphic to each other.

(b) For every Hopf surface S, we have

$$H^1(S, \mathcal{O})\cong H^1(S, \boldsymbol{C})\cong\boldsymbol{C},$$
$$H^1(S, \mathcal{O}^*)\cong H^1(S, \boldsymbol{C}^*)\cong\boldsymbol{C}^*.$$

The second isomorphism implies that every complex line bundle over S is flat. In particular every line bundle over $S_{\alpha_1\alpha_2\lambda}$ has the following form:

(1.1) $p: L(c)\to S_{\alpha_1\alpha_2\lambda}$, $L(c)=\boldsymbol{C}\times(\boldsymbol{C}^2-O)/\langle h\rangle$, where h denotes the group of automorphisms of $\boldsymbol{C}\times(\boldsymbol{C}^2-O)$ generated by $h(s, z_1, z_2)=(cs, \alpha_1 z_1+\lambda z_2^m, \alpha_2 z_2)$, $c\in\boldsymbol{C}^*$ ((s, z_1, z_2) denote the standard coordinates of $\boldsymbol{C}^3$) and the projection p is defined by $p([s, z_1, z_2])=([z_1, z_2])$ ([ ] denotes the class in the quotient space).

Definition 1.2. Let $L=L(c)$ be a complex line bundle over a primary Hopf surface $S=S_{\alpha_1\alpha_2\lambda}$. L is said to be of infinite type if there exists no triple of integers (p, q, r) such that $c^r=\alpha_1^p\alpha_2^q$ and either $p, q\geqq 0$, $r<0$ or $p, q\geqq 1$, $r\geqq 1$. Furthermore if there exists no pair of integers (p, q, r) such that $c^r=\alpha_1^p\alpha_2^q$, $p\geqq -1$, $q\geqq 0$, $r<0$ or $p\geqq 0$, $q\geqq -1$, $r<0$, or $p\geqq 1$, $q\geqq 1$, $r>0$, then L is said to be of strongly infinite type. We denote by $|L|$ the number $|c|$.

Our theorems are stated as follows.

Theorem 1. *Let S be a primary Hopf surface imbedded in a complex manifold M of dimension 3 and let N be the normal bundle of S. If N is of infinite type and $|N|<1$, then there exists a multiplicative holomorphic function u defined on some neighbourhood of S with divisor S.*

Theorem 2. *Let S be a primary Hopf surface imbedded in a complex manifold M of dimension 3. Suppose that the following conditions are satisfied.*

(1) *The normal bundle N of S is of strongly infinite type and $|N|\neq 1$.*

(2) *$[S]$ is a flat line bundle on some neighbourhood of S in M. Then there exists a tubular neighbourhood of S in M which is biholomorphic to a tubular neighbourhood of the 0-section of N.*

Theorem 3. *Let S be a primary Hopf surface imbedded in a complex manifold M of dimension 3. Suppose that the following condition is satisfied.*

($*$) *The normal bundle N of S is of strongly infinite type and $|N|<1$.*

Then there exists a tubular neighbourhood of S on M which is biholomorphic to a tubular neighbourhood of the 0-section of N.

Clearly Theorem 3 follows from Theorems 1 and 2.

2. Sketch of proofs. Because the proofs of Theorems 1 and 2 are similar we only sketch the proof of Theorem 1. Let S, N and M be the same as in Theorem 1. We divide the proof of Theorem 1 into three steps.

Step 1. First we construct special Stein coverings of S, $\mathcal{U}=\{U_i\}_{i=1}^6$ and $\mathcal{U}^*=\{U_i^*\}_{i=1}^6$ satisfying the following conditions.

(2.1) (1) Every U_i (or U_i^*) is biholomorphic to a Reinhaldt domain in $\boldsymbol{C}^2$. U_3 and U_6 contain, respectively, the Silov boundaries of U_4^*, U_5^*, U_6^* and of U_1^*, U_2^*, U_3^*. (2) Each U_i^* contains U_i as a relatively compact subset. (3) $U_1\cap U_2\cap U_3=\phi$, $U_4\cap U_5\cap U_6=\phi$, $U_1^*\cap U_2^*\cap U_3^*=\phi$, $U_4^*\cap U_5^*\cap U_6^*=\phi$. (4) Let U_{ijk} (or U_{ijk}^*) be a complex manifold obtained by gluing the disjoint union of U_i, U_j, U_k (or U_i^*, U_j^*, U_k^*) naturally on

$U_i \cap U_j$ and $U_j \cap U_k$ (or $U_i^* \cap U_j^*$, $U_j^* \cap U_k^*$) for $(i, j, k)=(1,2,3), (4,5,6)$. Then U_{123} and U_{456} (or U_{123}^*, U_{456}^*) are Stein manifolds. (5) Let W_{ij} be $(U_i^* \cap U_j) \cup (U_i \cap U_j^*)$ for $1 \leqq i < j \leqq 6$. Then every holomorphic function defined on w_{ij} extends to a holomorphic function defined on a domain W_{ij}^* $(\subset U_i^* \cap U_j^*)$ which contains $U_i \cap U_j$ as a relatively compact subset $(1 \leqq i < j \leqq 6)$, except for $(i, j)=(1,2)$, $(2,3)$, $(4,5)$, $(5,6)$. (For $(i, j)=(1,2), (2,3), (4,5), (5,6)$, W_{ij}^* is not determined naturally.) (6) There exist holomorphic vector fields Z_1, Z_2 defined on S such that zero loci of $\det(Z_1, Z_2) \in H^0(S, \Theta \wedge \Theta)$ do not intersect the Silov boundary of U_i for every i.

To construct such coverings, we use logarithmic convexity of the domain of convergence of a Laurent power series ([2]). Next we construct a Stein covering $\mathcal{V}^*=\{V_i^*\}_{i=1}^6$ of S in M and coordinates $(z_i, w_i): V_i^* \to \boldsymbol{C}^3$ for each i satisfying the following conditions;

(2.2) (1) V_i^* is a Stein neighbourhood of U_i^*. (2) (z_i, w_i) are defined on the closure of V_i^*. (3) $z_i: V_i^* \to \boldsymbol{C}^2$ is an extension of the coordinate $z_i | U_i^*$ of U_i^* and satisfies $z_i(V_i^*)=z_i(U_i^*)$. (4) $(z_i, w_i) | V_i^* \cap V_j^* = (z_j, w_j) | V_i^* \cap V_j^*$ for $(i, j)=(1,2), (2,3), (4,5), (5,6)$. (5) w_i; $V_i^* \to \boldsymbol{C}$ is the defining equation of U_i^* in V_i^*, i.e., $U_i^*=\{p \in V_i^* \mid w_i(p)=0\}$. (6) w_i/w_j is holomorphic on $V_i^* \cap V_j^*$ and $t_{ij}=w_i/w_j | U_i^* \cap U_j^*$ is a locally constant function on $U_i^* \cap U_j^*$.

To construct such $\mathcal{V}^*=\{V_i^*\}$ and (z_i, w_i) $(1 \leqq i \leqq 6)$, we use a result of Y. T. Siu ([5]).

Step 2. To prove Theorem 1, we must construct a system of holomorphic functions $\{u_i\}_{i=1}^6$ defined respectively on neighbourhoods V_i' $(\subseteqq V_i^*)$ of U_i^* satisfying the conditions (i). Each u_i is of the form $u_i(p)=w_i(p)+$(terms of order $\geqq 2$) (ii) $u_i=t_{ij}u_j$ on $V_i' \cap V_j'$. We determine each u_i as an implicit function defined by the equation

(2.3) $w_i=f_i(z_i, u_i)=u_i+\sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^\nu$, where $f_i(z_i, u_i)$ is a power series in u_i whose coefficients $f_{i|\nu}(z_i)$ are holomorphic functions of the variable z_i. To construct f_i as a formal power series we use entirely the same method as in [6]. The ν-th obstruction $-h_{ij|\nu+1}$ to construct the formal power series is an element to $Z^1(\mathcal{U}^*, \mathcal{O}(N^{-\nu}))$ and $f_{i|\nu+1}$ is determined by the equation

(2.4) $f_{i|\nu+1}(z_i)-t_{ij}^{-\nu} f_{j|\nu+1}(z_j)=-h_{ij|\nu+1}(z_i)$ on $U_i^* \cap U_j^*$.

The following lemma completes the construction of the formal power series.

Lemma. $\dim H^1(S, \mathcal{O}(L^{-\nu}))=0$ *for* $\nu \in \boldsymbol{Z}^+$ *if* L *is a complex line bundle of infinite type over* S.

Step 3. To prove that each f_i has a positive radius of convergence, we estimate $f_{i|\nu}$ by $f_{i|2} \cdots f_{i|\nu-1}$. Our estimate proceeds as follows.

(1) Estimate of $-h_{ij|\nu}$ on W_{ij}.

(2) Estimate of $f_{i|\nu}$ on U_i.

(3) Estimate of $f_{i|\nu}$ on U_i^*.

(1) is the estimate of the same type as in [6]. But we use a special norm on $Z^1(\mathcal{U}^*, \mathcal{O}(N^{-\nu}))$. (2) is obtained from (1) by using a similar method to in [1] and (2.2) (5). We note that $\{-h_{ij|\nu}\}=0$ for $(i, j)=(1,2)$, $(2,3)$, $(4,5)$, $(5,6)$ by the construction of coordinates. (3) is obtained from the equation (2.4) and arguments on the Silov boundary of U_i^*.

Using these estimates we can prove that each f_i has a positive radius of convergence.

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