83. A Remark on the Boundary Behavior of Quasiconformal Mappings and the Classification of Riemann Surfaces

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1. Generally, a property of open Riemann surfaces is not always preserved by a quasiconformal mapping. For example, the class O_{AB}, the class of Riemann surfaces on which there exists no non-constant bounded analytic function, is not quasicomformally invariant (cf. [1], [3]). In this paper, we shall study properties of Riemann surfaces which are not preserved by quasiconformal mappings.

Let R_1, R_2 be open Riemann surfaces and $f: R_1 \to R_2$ be a quasiconformal mapping. The main purpose of this paper is to construct the counter examples for the following problems.

I. Suppose that R_j $(j=1,2)$ are hyperbolic, that is, R_j have Green's functions $g_j(\cdot, p_j)$ with poles at $p_j \in R_j$. Are the Green's functions *quasi-invariant*? Precisely, dose the following inequality

$$g_1(z, p_1) \leqq M g_2(f(z), f(p_1))$$

hold for any point z on R_1 and a constant $M(>0)$ not depending on z?

II. Suppose R_1 is in *Widom class* (cf. [5]), that is, R_1 is hyperbolic and for each point $p_1 \in R_1$,

$$\int_0^\infty \beta(t: p_1)dt < +\infty,$$

where $\beta(t: p_1)$ is the first Betti number of $\{p \in R_1 : g_1(p, p_1) > t\}$. Is R_2 also in *Widom class*?

III. Let R_1 and R_2 be not in O_{AB}. Suppose that R_1 is *AB-separable*, that is, for any points $p, q \in R_1$ $(p \neq q)$ there is a bounded analytic function g such that $g(p) \neq g(q)$. Is R_2 also *AB-separable*?

Finally in § 4, we shall give a theorem concerning with Problems II and III.

2. First of all, we recall the following proposition due to A. Beurling and L. Ahlfors (cf. [1], [2]).

Proposition. *There exists a quasiconformal automorphism of the upper half plane with the boundary function $h(x)$ $(x \in \boldsymbol{R})$ if and only if*

$$\rho^{-1} \leqq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqq \rho \tag{1}$$

for some constant $\rho \geqq 1$ and for all x and $t(\neq 0)$.

Actually, if (1) *is satisfied there exists a mapping whose maximal dilatation*$\leqq\rho^2$. *For instance, this mapping is given by*

$$(2)\qquad \tilde{f}(z)=\frac{1}{2y}\int_{-y}^{y}h(x+s)ds+i\frac{r_h}{2y}\int_0^y(h(x+s)-h(x-s))ds$$

with $z=x+iy$, $y>0$ *and a certain constant* $r_h>0$.

We consider a function $h(x)=x^3$ on the real axis. It is easy to show that $h(x)$ satisfies (1) for some ρ. Hence $h(x)$ is the boundary function of a quasiconformal mapping $\tilde{f}$ defined by (2).

Since $\tilde{f}(iy)=ir_h y^3/4$, we can choose a sequence $\{y_n\}_1^\infty$ $(y_n>0)$ such that $\sum_{n=1}^\infty y_n=+\infty$ and $-i\sum_{n=1}^\infty \tilde{f}(iy_n)<+\infty$. Composing $\tilde{f}$ with a conformal mapping from the upper half plane onto the unit disk D, we verify that there are a quasiconformal automorphism F on D and a sequence $\{z_n\}_1^\infty$ $(|z_n|<1)$ such that

$$(3)\qquad \sum_{n=1}^\infty \log|z_n|=-\infty \quad\text{and}\quad \sum_{n=1}^\infty \log|F(z_n)|>-\infty.$$

Since $-\log|z|$ is the Green's function of D with a pole at the origin, this gives a counter example for Problem I.

Further, from (3) we have:

Corollary. *The zeros of a bounded analytic function on D are not preserved by a quasiconformal mapping.*

3. To construct a counter example for Problems II and III, we take a sequence $\{z_n\}_1^\infty$ $(0<z_n<1,\ n=1,2,\cdots)$ satisfying the condition (3). Put $W=D-\bigcup_{n=1}^\infty[z_{2n-1},z_{2n}]$, and we construct a two-sheeted covering surface R_2 from two copies W_1, W_2 of W, by identifying the upper and the lower edges crosswise along $\bigcup_{n=1}^\infty[z_{2n-1},z_{2n}]$. And we consider a quasiconformal mapping $\hat{F}$ on R_2 whose projection is F in § 2. Put $\hat{F}(R_2)=R_1$ and $\hat{F}^{-1}=f$, then R_1 is also a *two*-sheeted covering surface.

On the other hand, from (3) and a theorem of C. M. Stanton [4] R_1 is in *Widom class* and *AB-separable* but R_2 is *not* in *Widom class* and *not AB-separable*. Hence (R_1, R_2, f) is a desired counter example for Problems II and III.

4. For each $t>0$ we consider $h_t(x)=x|x|^t$. Then $h_t(x)$ satisfies (1) for some ρ_t, and we can take $1\leqq\rho_t\leqq(\sqrt{2}+1)^{2t}$ (cf. [2, p. 133]). Therefore, from Proposition in § 2, we can find a sequence $\{\tilde{f}_t\}_{t>0}$ of quasiconformal automorphisms of the upper half plane such that $\lim_{t\searrow 0}K(\tilde{f}_t)=1$ where $K(\tilde{f}_t)$ is a maximal dilatation of $\tilde{f}_t$. And $\tilde{f}_t$ is defined by (2) with h_t instead of h and with r_t instead of r_h.

Then we have $\tilde{f}_t(iy)=ir_t y^{1+t}/(2+t)$. Hence by the same argument as in §§ 2 and 3, we have the following:

Theorem. *There exist a sequence* $\{R_t\}_{t\geqq 0}$ *of Riemann surfaces and quasiconformal mappings* $f_t: R_0\to R_t$ *with* $\lim_{t\searrow 0}K(f_t)=1$ *such that* R_0 *is not in Widom class and not AB-separable, but all* $R_t(t>0)$ *are in Widom class and AB-separable.*

References

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