78. On the Regularity of Arithmetic Multiplicative Functions. III

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We present some new results concerning multiplicative functions.

1. Statement of results. Theorem. *Let $f(n)$ be a multiplicative arithmetic function. Suppose that there exists a positive non-decreasing function $g(x)$ such that*

i) $\lim_{x\to\infty} g(dx)/g(x)=h(d)$ *exists for any* $d\in N$,

ii) $\limsup_{x\to\infty} \frac{1}{g(x)} \sum_{n\leqslant x} |f(n+1)-f(n)|=0.$

a) *If*

iii) $\limsup_{x\to\infty} \frac{1}{g(x)} \left|\sum_{n\leqslant x} f(n)\right|>0,$

then, $f(n)$ is completely multiplicative, and there exists $\lambda\geqslant -1$ such that $|f(n)|=n^\lambda$.

b) *If*

iii)′ $\lim_{x\to\infty} \frac{1}{g(x)} \sum_{n\leqslant x} f(n)=M$ *exists and* $M\neq 0$,

then there exists $\lambda\geqslant -1$ such that $f(n)=n^\lambda$.

2. Sketch of proof of the theorem. We deduce from assumptions i)–iii) by partial summation that, for any $d\in N$,

$$(*)\qquad \left|\sum_{\substack{n\leqslant x\\ d|n}} f(n)-\frac{1}{d}\sum_{n\leqslant x} f(n)\right|=o(g(x)).$$

We can prove easily from here that $f(n)\neq 0$ for any $n\in N$. In fact, for any prime p and any positive integer r, we have

$$\left|f(p^r)\sum_{\substack{n\leqslant xp^{-r}\\ (n,p)=1}} f(n)-\left(1-\frac{1}{p}\right)\frac{1}{p^r}\sum_{n\leqslant x} f(n)\right|$$
$$=\left|\sum_{\substack{n\leqslant x\\ p^r|n}} f(n)-\frac{1}{p^r}\sum_{n\leqslant x} f(n)-\sum_{\substack{n\leqslant x\\ p^{r+1}|n}} f(n)+\frac{1}{p^{r+1}}\sum_{n\leqslant x} f(n)\right|=o(g(x)).$$

On the other hand, condition iii) gives

$$\limsup_{x\to\infty} \frac{1}{g(x)}\left(1-\frac{1}{p}\right)\frac{1}{p^r}\left|\sum_{n\leqslant x} f(n)\right|>0,$$

and consequently $f(p^r)\neq 0$ for any p and any r. Then we can prove the *complete multiplicativity* of $f(n)$, by means of the same method as

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in [1]. Let q be an even positive integer, k an integer $\geqslant 2$, and put $S_k(q)=q^{k-1}+\cdots+1$. We obtain, similarly as in [1], that

$$\lim_{x\to\infty}|f(q^k)-f(q)^k|\cdot|f(S_k(q))|\cdot\frac{1}{g(x)}\left|\sum_{n\leqslant x}f(S_k(q)qn)\right|=0\,;$$

since $f(S_k(q))\neq 0$ and

$$\limsup_{x\to\infty}\frac{1}{g(x)}\left|\sum_{n\leqslant x}f(S_k(q)qn)\right|$$

$$=\limsup_{x\to\infty}\frac{1}{g(x)}\,\frac{1}{S_k(q)q}\left|\sum_{n\leqslant S_k(q)qx}f(n)\right|>0,$$

we get $f(q^k)=f(q)^k$, and this implies $f(p^k)=f(p)^k$ for any prime p and any $k\geqslant 0$.

If $f(n)$ satisfies iii), then so does $|f(n)|$ and this is a *positive completely multiplicative* function. Then the formula (*) gives:

$$\left|\frac{S(x/d)}{S(x)}-\frac{1}{d\,|f(d)|}\right|=\frac{o(g(x))}{g(x)}\,\frac{g(x)}{S(x)},$$

where $S(x)=\sum_{n\leqslant x}|f(n)|$. Using the condition iii), we can prove from here that $1/d\,|f(d)|$ is a positive non-decreasing multiplicative function, which gives a).

Now, since the condition iii)$'$ is a stronger assumption than iii), $f(n)$ is a completely multiplicative function. Put

$$M(x)=\frac{1}{g(x)}\sum_{n\leqslant x}f(n),$$

then (*) gives

$$\left|f(d)\cdot M(x)-\frac{h(d)}{d}\cdot M(dx)\right|=o(1).$$

From iii)$'$ follows now $f(d)=h(d)/d$. On the other hand, $h(d)$ is clearly a positive non-decreasing multiplicative function. So $h(d)=d^{\lambda+1}$ for some $\lambda\geqslant -1$, and we get b).

Reference

[1] J.-L. Mauclaire and L. Murata: On the regularity of arithmetic multiplicative functions. I. Proc. Japan Acad., **56A**, 438–440 (1980).