73. *On Cayley-Aronhold Realizations of* sl$(n+1, K)$

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(Communicated by Kunihiko KODAIRA, M. J. A., June 11, 1981)

1. In the present note, we shall sketch an outline of invariant theory of formal power series in $(r+1)\times(n-r)$-variable matrix $z=(z_{\alpha i})_{0\leq\alpha\leq r, r+1\leq i\leq n}$ with respect to sl $(n+1, K)$.

First, we shall list notations freely used:

K: a fixed field of characteristic zero,

n: a positive integer,

r: a non-negative integer satisfying $0\leq r\lneqq n$,

$\alpha, \beta, \gamma, \cdots, \alpha', \beta', \gamma', \cdots$ run over $\{0, 1, 2, \cdots, r\}$,

$a, b, c, \cdots, a', b', c', \cdots$ run over $\{r+1, r+2, \cdots, n\}$,

$$z=\begin{pmatrix} z_{0r+1}, & \cdots, & z_{0n} \\ \vdots & & \vdots \\ z_{rr+1}, & \cdots, & z_{rn} \end{pmatrix}: \text{ a variable } (r+1)\times(n-r)\text{-matrix},$$

$\varepsilon_{\alpha a}$: the specialization of z such that

$$z_{\beta b}\longmapsto\begin{cases}1 & (\beta, b)=(\alpha, a),\\ 0 & (\beta, b)\neq(\alpha, a),\end{cases}$$

$$\mathcal{L}=\left\{l=\begin{pmatrix} l_{0r+1}, & \cdots, & l_{0n} \\ \vdots & & \vdots \\ l_{rr+1}, & \cdots, & l_{rn} \end{pmatrix}\middle| l_{\alpha a} \text{ run over non-negative integers}\right\},$$

$$z^l=\prod_{\alpha,a} z_{\alpha a}^{l_{\alpha a}},$$

$$\left(\frac{\partial}{\partial z}\right)^l=\prod_{\alpha,a}\left(\frac{\partial}{\partial z_{\alpha a}}\right)^{l_{\alpha a}},$$

$$l\,!=\prod_{\alpha,a} l_{\alpha a}\,!, \qquad \sum l=\sum_{\alpha,a} l_{\alpha a},$$

$e_{\alpha\beta}, e_{\alpha a}, e_{a\alpha}, e_{ab}$: the $(n+1)\times(n+1)$-matrices whose only non-zero entries are, respectively, the $(\alpha, \beta), (\alpha, a), (a, \alpha), (a, b)$-entries with value one.

Remark.

$$(l+\varepsilon_{\alpha a})\,!=l\,!\,(l_{\alpha a}+1)$$

$$(l+\varepsilon_{\alpha a}+\varepsilon_{\beta b})\,!=\begin{cases}l\,!\,(l_{\alpha a}+1)(l_{\alpha a}+2) & (\alpha, a)=(\beta, b),\\ l\,!\,(l_{\alpha a}+1)(l_{\beta b}+2) & (\alpha, a)\neq(\beta, b),\end{cases}$$

$$\sum(l+\varepsilon_{\alpha a})=\sum l+1, \quad \sum(l+\varepsilon_{\alpha a}-\varepsilon_{\beta b})=\sum l \quad (l_{\beta b}\geq 1).$$

2. We denote by $\xi=(\xi^{(l)})_{l\in\mathcal{L}}$ a vector of infinite length with indeterminate entries, and choose an element $w\neq 0, 1, 2, \cdots$ in K. The basic formal power series is defined by

$$f(\xi \mid z)=\sum_{l\in\mathcal{L}} \frac{(w)_l}{l!}\xi^{(l)}z^l,$$

where

$$(w)_l=w(w-1)\cdots(w-\sum l+1).$$

Two systems of operators acting on polynomial algebras $K[\xi]$ and $K[z]$ are defined by

$$(1)\quad \begin{cases} D_{\alpha\beta}=\sum_c z_{\beta c}\dfrac{\partial}{\partial z_{\alpha c}}, \\ D_{\alpha a}=\dfrac{\partial}{\partial z_{\alpha a}}, \\ D_{a\alpha}=wz_{\alpha a}-\sum_{\gamma,c} z_{\alpha c}z_{\gamma\alpha}\dfrac{\partial}{\partial z_{\gamma c}}, \\ D_{ab}=-\sum_{\gamma} z_{\gamma a}\dfrac{\partial}{\partial z_{\gamma b}}, \\ H_{\alpha a}=w-\sum_c z_{\alpha c}\dfrac{\partial}{\partial z_{\alpha c}}-\sum_{\gamma} z_{\gamma a}\dfrac{\partial}{\partial z_{\gamma a}}=w-D_{\alpha\alpha}+D_{aa}, \end{cases}$$

$$(2)\quad \begin{cases} \mathcal{E}_{\alpha\beta}=\sum_l \left(\sum_c l_{\beta c}\xi^{(l-\varepsilon_{\beta c}+\varepsilon_{\alpha c})}\right)\dfrac{\partial}{\partial \xi^{(l)}} \qquad (\alpha\neq\beta), \\ \Delta_{\alpha a}=\sum_l (w-\sum l)\xi^{(l+\varepsilon_{\alpha a})}\dfrac{\partial}{\partial \xi^{(l)}}, \\ \mathcal{D}_{a\alpha}=\sum_l (w-\sum l+1)^{-1}\Big\{l_{\alpha a}\Big(w-\sum_c l_{\alpha c}-\sum_{\gamma} l_{\gamma a}+l_{\alpha a}+1\Big)\xi^{(l-\varepsilon_{\alpha a})} \\ \qquad\qquad -\sum_{\substack{\gamma\neq a\\ c\neq a}} l_{\alpha c}l_{\gamma a}\xi^{(l-\varepsilon_{\alpha c}-\varepsilon_{\gamma a}+\varepsilon_{\gamma c})}\Big\}\dfrac{\partial}{\partial \xi^{(l)}}, \\ \mathcal{F}_{ab}=-\sum_l \left(\sum_{\gamma} l_{\gamma a}\xi^{(l-\varepsilon_{\gamma a}+\varepsilon_{\gamma b})}\right)\dfrac{\partial}{\partial \xi^{(l)}} \qquad (a\neq b), \\ \mathcal{H}_{\alpha a}=\sum_l \left(w-\sum_c l_{\alpha c}-\sum_l l_{\gamma a}\right)\xi^{(l)}\dfrac{\partial}{\partial \xi^{(l)}}. \end{cases}$$

By simple calculation we have the following

Lemma 1.

$$(3)\quad \begin{cases} \mathcal{E}_{\alpha\beta}f(\xi \mid z)=D_{\alpha\beta}f(\xi \mid z) \qquad (\alpha\neq\beta), \\ \Delta_{\alpha a}f(\xi \mid z)=D_{\alpha a}f(\xi \mid z), \\ \mathcal{D}_{a\alpha}f(\xi \mid z)=D_{a\alpha}f(\xi \mid z), \\ \mathcal{F}_{ab}f(\xi \mid z)=D_{ab}f(\xi \mid z) \qquad (a\neq b), \\ \mathcal{H}_{\alpha a}f(\xi \mid z)=H_{\alpha a}f(\xi \mid z). \end{cases}$$

Lemma 2. *The mapping*

$$(4)\quad \begin{cases} e_{\alpha\beta}\longmapsto D_{\alpha\beta} \qquad (\alpha\neq\beta), \\ e_{\alpha a}\longmapsto D_{\alpha a}, \\ e_{a\alpha}\longmapsto D_{a\alpha}, \\ e_{ab}\longmapsto D_{ab} \qquad (a\neq b), \\ -e_{\alpha\alpha}+e_{aa}\longmapsto H_{\alpha a} \end{cases}$$

induces an anti-realization, anti-isomorphism, of sl $(n+1, K)$.

Since the operators (2) commute with the operators (3), by virtue of Lemmas 1, 2, we have the following

Theorem 1. *The mapping*

$$(5)\quad \begin{cases} e_{\alpha\beta}\longmapsto \varepsilon_{\alpha\beta} & (\alpha\neq\beta),\\ e_{\alpha a}\longmapsto \Delta_{\alpha a}, \\ e_{a\alpha}\longmapsto \mathcal{D}_{a\alpha}, \\ e_{ab}\longmapsto \mathcal{F}_{ab} & (a\neq b),\\ -e_{\alpha\alpha}+e_{aa}\longmapsto \mathcal{H}_{\alpha a} \end{cases}$$

induces a realization, and isomorphisms, of sl $(n+1, K)$.

We call this isomorphism Cayley-Aronhold realization for $f(\xi\,|\,z)$.

Remark. For special case $r=0$, denoting $z_{0a}=z_a$, $l_{0a}=l_a$, $\varepsilon_{0a}=\varepsilon_a$, $\Delta_{0a}=\Delta_a$, $\mathcal{D}_{a0}=\mathcal{D}_a$, $\mathcal{H}_{0a}=\mathcal{H}_a$, we have simple expression:

$$(6)\quad \begin{cases} \mathcal{D}_a=\sum_l l_a \xi^{(l-\varepsilon_a)}\dfrac{\partial}{\partial\xi^{(l)}},\\ \Delta_a=\sum_l (w-\sum l)\xi^{(l+\varepsilon_a)}\dfrac{\partial}{\partial\xi^{(l)}},\\ \mathcal{H}_a=\sum_l (w-\sum l-l_a)\xi^{(l)}\dfrac{\partial}{\partial\xi^{(l)}},\\ \mathcal{F}_{ab}=-\sum_l l_a\xi^{(l-\varepsilon_a+\varepsilon_b)}\dfrac{\partial}{\partial\xi^{(l)}} & (a\neq b). \end{cases}$$

Remark. For $n=\infty$ we may use the same expression of Cayley-Aronhold operators (2).

3. Before defining semi-invariants and covariants, we define Cayley-Aronhold operators to several basic formal power series

$$\begin{cases} f_1(\xi_1\,|\,z)=\sum_l \dfrac{(w_1)_l}{l!}\xi_1^{(l)}z^l,\\ \quad\vdots\\ f_N(\xi_N\,|\,z)=\sum_l \dfrac{(w_N)_l}{l!}\xi_N^{(l)}z^l, \end{cases}$$

as follows

$$\mathcal{E}_{\alpha\beta}=\sum_{s=1}^N \mathcal{E}_{\alpha\beta}^{(s)},\qquad \Delta_{\alpha a}=\sum_{s=1}^N \Delta_{\alpha a}^{(s)},\qquad \mathcal{D}_{a\alpha}=\sum_{s=1}^N \mathcal{D}_{a\alpha}^{(s)},$$
$$\mathcal{F}_{ab}=\sum_{s=1}^N \mathcal{F}_{ab}^{(s)},\qquad \mathcal{H}_{\alpha a}=\sum_{s=1}^N \mathcal{H}_{\alpha a}^{(s)},$$

where $\mathcal{E}_{\alpha\beta}^{(s)}$, $\Delta_{\alpha a}^{(s)}$, $\mathcal{D}_{a\alpha}^{(s)}$, $\mathcal{F}_{ab}^{(s)}$, $\mathcal{H}_{\alpha a}^{(s)}$ are Cayley-Aronhold operators for $f_s(\xi_s\,|\,z)$.

Definition 1. Elements of the subalgebra

$$\mathfrak{S}=\{\varphi\in K[\xi]\,|\,\mathcal{E}_{\alpha\beta}\varphi=\mathcal{D}_{a\alpha}\varphi=\mathcal{F}_{ab}\varphi=0 \qquad (\alpha\neq\beta,\ a\neq b)$$

are called semi-invariants of $(f_1(\xi_1\,|\,z),\cdots,f_N(\xi_N\,|\,z))$.

Proposition 1.

$$(7)\quad \mathfrak{S}=\bigoplus_u \mathfrak{S}^{[u]},\qquad \mathfrak{S}^{[u]}=\{\varphi\in\mathfrak{S}\,|\,\mathcal{H}_{\alpha i}\varphi=u\varphi\ (0\leq\alpha\leq r,\ r+1\leq a\leq n)\}.$$

This is a consequence of the relations

$$\mathcal{H}_{\alpha a}-\mathcal{H}_{\beta a}=-[\mathcal{E}_{\alpha\beta},\mathcal{E}_{\beta\alpha}] \qquad (\alpha\neq\beta),$$
$$\mathcal{H}_{\beta a}-\mathcal{H}_{\beta b}=-[\mathcal{F}_{ab},\mathcal{F}_{ba}] \qquad (a\neq b),$$
$$\mathcal{H}_{\alpha a}-\mathcal{H}_{\beta b}=-[\mathcal{E}_{\alpha\beta},\mathcal{E}_{\beta\alpha}]+[\mathcal{F}_{ab},\mathcal{F}_{ba}] \qquad (\alpha\neq\beta,\ a\neq b).$$

Definition 2. A covariant of index u for $(f_1(\xi_1|z),\cdots,f_N(\xi_N|z))$ is a differential polynomial of $f_s(\xi_s|z)$ $(1\leq s\leq N)$ given by

$$\varphi\left(\cdots,\frac{(\partial/\partial z)^l f_s(\xi_s|z)}{(w_s)_l},\cdots\right) \tag{8}$$

with a semi-invariant $\varphi(\cdots,\xi_s^{(l)},\cdots)$ in $\mathfrak{S}^{[u]}$.

Tayler expansions of (8) are given by

$$\exp\left(\sum_{\alpha,a} z_{\alpha a}\Delta_{\alpha a}\right)\varphi(\cdots,\xi_l^{(s)},\cdots)=\sum_{l\in\mathcal{L}}\frac{1}{l!}\Delta^l\varphi(\cdots,\xi_l^{(s)},\cdots)z^l, \tag{9}$$

where

$$\Delta^l=\prod_{\alpha,a}\Delta_{\alpha a}^{l_{\alpha a}}.$$

References

[1] H. Morikawa: Invariant Theory. Kinokuniya (1977) (in Japanese).
[2] ——: Some analytic and geometric applications of the invariant theoretic method. Nagoya Math. J., **80**, 1–47 (1980).
[3] R. A. Rankin: The construction of automorphic forms from the derivatives of a given form. J. Indian Math. Soc., **20**, 103–116 (1956).
[4] I. Schur: Vorlesungen über Invariantentheorie. Springer (1968).