67. On the Isomonodromic Deformation for Linear Ordinary Differential Equations of the Second Order

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§ 1. **Introduction.** It was R. Fuchs ([1]) who gave first an example of the isomonodromic deformation. Considering the differential equation of Fuchsian type

$$\frac{d^2y}{dx^2}=\left(\frac{a}{x^2}+\frac{b}{(x-1)^2}+\frac{c}{(x-t)^2}+\frac{3}{4(x-\lambda)^2}\right.$$
$$\left.+\frac{\rho}{x}+\frac{\sigma}{x-1}+\frac{\tau}{x-t}+\frac{\omega}{x-\lambda}\right)y$$

having $x=\lambda$ as apparent singularity, he rediscovered the sixth Painlevé equation as isomonodromic deformation equation. Then R. Garnier ([2]) derived all the other Painlevé equations by the isomonodromic deformation for linear differential equations of the form

$$y''=py \tag{1.1}$$

with irregular singularities and an apparent singularity. (For the isomonodromic deformation of equations with irregular singularities, see [3], [4], [7].)

Recently K. Okamoto ([5], [6]) found the following two remarkable facts: 1) The Painlevé equations are converted into Hamiltonian systems, called the Painlevé systems, with polynomial Hamiltonian functions. 2) If the linear differential equations considered by Fuchs and Garnier are transformed into equations of the form

$$y''+p_1y'+p_2y=0 \tag{1.2}$$

in a canonical way, then the isomonodromic deformation for the transformed equations is governed by the Painlevé systems.

Fuchs and Garnier, and hence Okamoto supposed that the difference of the exponents at the apparent singularity is just two up to the sign. The purpose of this note is to discuss the case when this difference is greater than two.

§ 2. **Preliminaries.** If the equation (1.2) is transformed into an equation of the form (1.1), we have

$$p=\frac{1}{2}p_1'+\frac{1}{4}p_1^2-p_2. \tag{2.1}$$

Suppose that p_1 and p_2 are rational in x and in several parameters.

We consider the case when just one of the parameters, say t, can be taken as a deformation parameter and the other parameters are viewed as functions of t. It is known that the equation (1.2) can be deformed in an isomonodromic way, if and only if there exist linearly independent solutions $y_1(x,t)$, $y_2(x,t)$ depending analytically on t of the equation (1.2) and two functions $A(x,t)$, $B(x,t)$ rational in x and analytic in t which satisfy

$$\partial y_j/\partial t = A\partial y_j/\partial x + By_j \qquad (j=1,2).$$

The existence of y_1, y_2, A, B implies that the system of equations

$$\begin{cases} \partial y/\partial x = z \\ \partial z/\partial x = -p_1 z - p_2 y \\ \partial y/\partial t = Az + By \end{cases}$$

is completely integrable, whence we have

$$\partial^2 B/\partial x^2 + p_1 \partial B/\partial x - 2p_1 \partial A/\partial x - (\partial p_2/\partial x)A + \partial p_2/\partial t = 0,$$
$$2\partial B/\partial x + \partial^2 A/\partial x^2 - p_1 \partial A/\partial x - (\partial p_1/\partial x)A + \partial p_1/\partial t = 0.$$

Eliminating B, we obtain

(2.2) $$2^{-1}\partial^3 A/\partial x^3 - 2p\partial A/\partial x - (\partial p/\partial x)A + \partial p/\partial t = 0,$$

where p is given by (2.1). It follows that the equation (1.1) can be deformed in an isomonodromic way, if and only if there exists a function $A(x,t)$ which is rational in x, analytic in t and satisfies (2.2). We remark that we obtain from (2.1)

(2.3) $$A = \begin{vmatrix} y_1 & \partial y_1/\partial t \\ y_2 & \partial y_2/\partial t \end{vmatrix} \div \begin{vmatrix} y_1 & \partial y_1/\partial x \\ y_2 & \partial y_2/\partial x \end{vmatrix}.$$

§ 3. **Linear differential equations.** For each positive integer n, we consider the following six linear differential equations

L_{VI}^{n}: $$y'' + \left(\frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \frac{1-\theta}{x-t} - \frac{n}{x-\lambda}\right)y' + \left(\frac{\kappa}{x(x-1)} - \frac{t(t-1)H}{x(x-1)(x-t)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}\right)y = 0,$$

L_{V}^{n}: $$y'' + \left(\frac{1-\kappa_0}{x} + \frac{\eta_1 t}{(x-1)^2} + \frac{2-\theta}{x-1} - \frac{n}{x-\lambda}\right)y' + \left(\frac{\kappa}{x(x-1)} - \frac{tH}{x(x-1)^2} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}\right)y = 0,$$

L_{IV}^{n}: $$y'' + \left(\frac{1-\kappa_0}{x} - \frac{x}{2} - t - \frac{n}{x-\lambda}\right)y' + \left(\frac{\theta}{2} - \frac{H}{2x} + \frac{\lambda\mu}{x(x-\lambda)}\right)y = 0,$$

L_{III}^{n}: $$y'' + \left(\frac{\eta_0 t}{x^2} + \frac{2-\theta_0}{x} - \eta_\infty t - \frac{n}{x-\lambda}\right)y' + \left(\frac{\eta_\infty(\theta_0+\theta_\infty)}{2x} - \frac{tH+\lambda\mu}{2x^2} + \frac{\lambda\mu}{x(x-\lambda)}\right)y = 0,$$

L_{II}^{n}: $$y'' - \left(2x^2 + t + \frac{n}{x-\lambda}\right)y' + \left(-2\theta x - 2H + \frac{\mu}{x-\lambda}\right)y = 0,$$

$$\mathrm{L}^n_{\mathrm{I}}:\quad y''-\frac{n}{x-\lambda}y'+\left(-4x^3-2tx-2H+\frac{\mu}{x-\lambda}\right)y=0.$$

The Riemann schemes R^n_J for these equations are given as follows:

$$\mathrm{R}^n_{\mathrm{VI}}\quad\left\{\begin{array}{ccccc} x=0 & x=1 & x=t & x=\lambda & x=\infty \\ 0 & 0 & 0 & 0 & \chi \\ \kappa_0 & \kappa_1 & \theta & n+1 & \chi+\kappa_\infty \end{array}\right\},$$

where $\kappa=((\kappa_0+\kappa_1+\theta+n-2)^2-\kappa_\infty^2)/4=\chi(\chi+\kappa_\infty)$,

$$\mathrm{R}^n_{\mathrm{V}}\quad\left\{\begin{matrix} x=0 \\ 0 \\ \kappa_0 \end{matrix}\quad \begin{matrix} x=1 \\ \overbrace{0\quad 0} \\ \eta_1 t\quad \theta \end{matrix}\quad \begin{matrix} x=\lambda \\ 0 \\ n+1 \end{matrix}\quad \begin{matrix} x=\infty \\ \chi \\ \chi+\kappa_\infty \end{matrix}\right\},$$

where $\kappa=((\kappa_0+\theta+n-2)^2-\kappa_\infty^2)/4=\chi(\chi+\kappa_\infty)$,

$$\mathrm{R}^n_{\mathrm{IV}}\quad\left\{\begin{matrix} x=0 \\ 0 \\ \kappa_0 \end{matrix}\quad \begin{matrix} x=\lambda \\ 0 \\ n+1 \end{matrix}\quad \overbrace{\begin{matrix} 0 & 0 & \theta \\ 1/4 & t & -\theta-\kappa_0-n+2 \end{matrix}}^{x=\infty}\right\},$$

$$\mathrm{R}^n_{\mathrm{III}}\quad\left\{\overbrace{\begin{matrix} 0 & 0 \\ \eta_0 t & \theta_0 \end{matrix}}^{x=0}\quad \begin{matrix} x=\lambda \\ 0 \\ n+1 \end{matrix}\quad \overbrace{\begin{matrix} 0 & -(\theta_\infty+\theta_0)/2 \\ \eta_\infty t & (\theta_\infty-\theta_0)/2-n+2 \end{matrix}}^{x=\infty}\right\},$$

$$\mathrm{R}^n_{\mathrm{II}}\quad\left\{\begin{matrix} x=\lambda \\ 0 \\ n+1 \end{matrix}\quad \overbrace{\begin{matrix} 0 & 0 & 0 & \theta \\ 2/3 & 0 & t & -\theta-n+2 \end{matrix}}^{x=\infty}\right\},$$

$$\mathrm{R}^n_{\mathrm{I}}\quad\left\{\begin{matrix} x=\lambda \\ 0 \\ n+1 \end{matrix}\quad \overbrace{\begin{matrix} -4/5 & 0 & 0 & 0 & -t & 3/2-n \\ 4/5 & 0 & 0 & 0 & t & 3/2-n \end{matrix}}^{x=\infty\ (1/2)}\right\}.$$

§4. **Confluence of singular points.** It was proved by Okamoto that the equations $\mathrm{L}^1_{\mathrm{V}}$–$\mathrm{L}^1_{\mathrm{I}}$ are derived from $\mathrm{L}^1_{\mathrm{VI}}$ by a process of step-by-step confluence of singularities. This result is generalized as follows.

Theorem 1. *The equations* $\mathrm{L}^n_{\mathrm{V}}$–$\mathrm{L}^n_{\mathrm{I}}$ *are derived from* $\mathrm{L}^n_{\mathrm{VI}}$ *by a process of step-by-step confluence of singularities in the following order:*

$$\begin{tikzcd} & & \mathrm{L}^n_{\mathrm{IV}} \arrow[dr] & & \\ \mathrm{L}^n_{\mathrm{VI}} \arrow[r] & \mathrm{L}^n_{\mathrm{V}} \arrow[ur] \arrow[dr] & & \mathrm{L}^n_{\mathrm{II}} \arrow[r] & \mathrm{L}^n_{\mathrm{I}}. \\ & & \mathrm{L}^n_{\mathrm{III}} \arrow[ur] & & \end{tikzcd}$$

The process of step-by-step confluence is carried out as follows:

$\mathrm{L}^n_{\mathrm{VI}}\longrightarrow\mathrm{L}^n_{\mathrm{V}}$: $x\longrightarrow x$, $\lambda\longrightarrow\lambda$, $\mu\longrightarrow\mu$, $t\longrightarrow 1+\varepsilon t$, $\kappa_0\longrightarrow\kappa_0$, $\kappa_1\longrightarrow\eta_1/\varepsilon+\theta$, $\kappa_\infty\longrightarrow\kappa_\infty$, $\theta\longrightarrow-\eta_1/\varepsilon$, $H\longrightarrow H/\varepsilon$ *and then* $\varepsilon\longrightarrow 0$.

$\mathrm{L}^n_{\mathrm{V}}\longrightarrow\mathrm{L}^n_{\mathrm{IV}}$: $x\longrightarrow\varepsilon x/\sqrt{2}$, $\lambda\longrightarrow\varepsilon\lambda/\sqrt{2}$, $\mu\longrightarrow\sqrt{2}\,\mu/\varepsilon$, $t\longrightarrow 1+\sqrt{2}\,\varepsilon t$, $\kappa_0\longrightarrow\kappa_0$, $\eta_1\longrightarrow-1/\varepsilon^2$, $\kappa_\infty\longrightarrow 1/\varepsilon^2$, $\theta\longrightarrow 1/\varepsilon^2+2\theta-\kappa_0-n+2$, $H+k/t\longrightarrow H/\sqrt{2}\,\varepsilon$ *and then* $\varepsilon\longrightarrow 0$,

$\mathrm{L}_{\mathrm{V}}^{n}\longrightarrow\mathrm{L}_{\mathrm{III}}^{n}$: $x\longrightarrow 1+\varepsilon tx$, $\lambda\longrightarrow 1+\varepsilon t\lambda$, $\mu\longrightarrow\mu/\varepsilon t$, $t\longrightarrow t^2$,
$\kappa_0\longrightarrow\eta_\infty/\varepsilon-n+1$, $\eta_1\longrightarrow\varepsilon\eta_0$, $\kappa_\infty\longrightarrow\eta_\infty/\varepsilon-\theta_\infty-1$,
$\theta\longrightarrow\theta_0$, $H\longrightarrow(H+\lambda\mu/t)/2t$ *and then* $\varepsilon\longrightarrow 0$,

$\mathrm{L}_{\mathrm{IV}}^{n}\longrightarrow\mathrm{L}_{\mathrm{II}}^{n}$: $x\longrightarrow\sqrt[3]{4}\,x/\varepsilon+1/\varepsilon^3$, $\lambda\longrightarrow\sqrt[3]{4}\,\lambda/\varepsilon+1/\varepsilon^3$, $\mu\longrightarrow\varepsilon\mu/\sqrt[3]{4}$,
$t\longrightarrow\varepsilon t/\sqrt[3]{4}+1/\varepsilon^3$, $\kappa_0\longrightarrow 1/2\varepsilon^6$, $\theta\longrightarrow-\theta$,
$H\longrightarrow\sqrt[3]{4}\,H/\varepsilon-\theta/\varepsilon^3$, *and then* $\varepsilon\longrightarrow 0$,

$\mathrm{L}_{\mathrm{III}}^{n}\longrightarrow\mathrm{L}_{\mathrm{II}}^{n}$: $x\longrightarrow 1+2\varepsilon x$, $\lambda\longrightarrow 1+2\varepsilon\lambda$, $\mu\longrightarrow\mu/2\varepsilon$, $t\longrightarrow 1+\varepsilon^2 t$,
$\eta_0\longrightarrow-1/4\varepsilon^3$, $\eta_\infty\longrightarrow 1/4\varepsilon^3$, $\theta_0\longrightarrow-1/2\varepsilon^2-2\theta$,
$\theta_\infty\longrightarrow 1/2\varepsilon^3$, $H\longrightarrow H/\varepsilon^{-2}$ *and then* $\varepsilon\longrightarrow 0$,

$\mathrm{L}_{\mathrm{II}}^{n}\longrightarrow\mathrm{L}_{\mathrm{I}}^{n}$: $y\longrightarrow y\exp(x^3/3+tx/2)$ *and then* $x\longrightarrow\varepsilon x+1/\varepsilon^5$,
$\lambda\longrightarrow\varepsilon\lambda+1/\varepsilon^5$, $\mu-n\lambda^2-nt/2\longrightarrow\mu/\varepsilon$, $t\longrightarrow\varepsilon^2 t-6/\varepsilon^{10}$,
$\theta\longrightarrow 4\varepsilon^{-15}+1-n/2$, $H+n\lambda+nt^2/2\longrightarrow H/\varepsilon^2-t/2\varepsilon^8-3/\varepsilon^{20}$
and finally $\varepsilon\longrightarrow 0$.

§5. **Hamiltonian systems.** We suppose that $x=\lambda$ is an apparent singularity for each equation L_J^n. Then we obtain a relation among t,λ,μ,H for each equation, and hence H can be considered as a function of t,λ,μ which is denoted by H_J^n. It is easy to see that H_J^1 $(J=\mathrm{IV},\cdots,\mathrm{I})$ are polynomials of λ,μ with rational coefficients in t, and H_J^2 $(J=\mathrm{IV},\cdots,\mathrm{I})$ are rational in λ,μ,t and that for $n\geqq 3$, H_J^n are algebraic in λ,μ,t.

We assign to each L_J^n the Hamiltonian system:

$$\mathrm{P}_J^n \qquad \begin{cases} d\lambda/dt=\partial H_J^n/\partial\mu \\ d\mu/dt=-\partial H_J^n/\partial\lambda. \end{cases}$$

From Theorem 1, we obtain the following theorem.

Theorem 2. *The systems* $\mathrm{P}_{\mathrm{V}}^n$–$\mathrm{P}_{\mathrm{I}}^n$ *are derived from* $\mathrm{P}_{\mathrm{VI}}^n$ *by the same process of replacements as in Theorem* 1 *except for the replacements of* x *in the order*

$$\begin{tikzcd}[row sep=small] & & \mathrm{P}_{\mathrm{IV}}^n \arrow[dr] & & \\ \mathrm{P}_{\mathrm{VI}}^n \arrow[r] & \mathrm{P}_{\mathrm{V}}^n \arrow[ur] \arrow[dr] & & \mathrm{P}_{\mathrm{II}}^n \arrow[r] & \mathrm{P}_{\mathrm{I}}^n. \\ & & \mathrm{P}_{\mathrm{III}}^n \arrow[ur] & & \end{tikzcd}$$

We transform the equations L_J^n into the following equations of the form (1.1):

$$\tilde{\mathrm{L}}_{\mathrm{VI}}^n:\quad y''=\Big(\frac{a}{x^2}+\frac{b}{(x-1)^2}+\frac{c}{(x-t)^2}+\frac{d}{x(x-1)}+\frac{n^2+2n}{4(x-\lambda)^2}+\frac{t(t-1)\tilde{H}}{x(x-1)(x-t)}-\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)}\Big)y,$$

$$\tilde{\mathrm{L}}_{\mathrm{V}}^n:\quad y''=\Big(\frac{a}{x^2}+\frac{bt^2}{(x-1)^4}+\frac{ct}{(x-1)^3}+\frac{d}{x(x-1)}+\frac{n^2+2n}{4(x-\lambda)^2}+\frac{t\tilde{H}}{x(x-1)^2}+\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)}\Big)y,$$

$$\tilde{\mathrm{L}}_{\mathrm{IV}}^n:\quad y''=\Big(\frac{a}{x^2}+\frac{x^2}{16}+\frac{tx}{4}+\frac{t^2}{4}+b+\frac{n^2+2n}{4(x-\lambda)^2}+\frac{\tilde{H}}{2x}-\frac{\lambda\nu}{x(x-\lambda)}\Big)y,$$

$$\tilde{\mathrm{L}}^n_{\mathrm{III}}:\quad y''=\Big(\frac{at^2}{x^4}+\frac{bt}{x^3}+\frac{ct}{x}+dt^2+\frac{n^2+2n}{4(x-\lambda)^2}+\frac{t\tilde{H}+\lambda\nu}{2x^2}-\frac{\lambda\nu}{x(x-\lambda)}\Big)y,$$

$$\tilde{\mathrm{L}}^n_{\mathrm{II}}:\quad y''=\Big(x^4+tx^2+ax+\frac{n^2+2n}{4(x-\lambda)^2}+2\tilde{H}-\frac{\nu}{x-\lambda}\Big)y,$$

$$\tilde{\mathrm{L}}_{\mathrm{I}}:\quad y''=\Big(4x^3+2tx+\frac{n^2+2n}{4(x-\lambda)^2}+2\tilde{H}-\frac{\nu}{x-\lambda}\Big)y.$$

Suppose that $x=\lambda$ is an apparent singularity for each $\tilde{\mathrm{L}}^n_J$. Then for each $\tilde{\mathrm{L}}^n_J$, $\tilde{H}$ can be considered as a function of t, λ, ν which we denote by $\tilde{H}^n_J$. We define a Hamiltonian system $\tilde{\mathrm{P}}^n_J$ by

$$\tilde{\mathrm{P}}^n_J:\qquad \begin{cases} d\lambda/dt=\partial\tilde{H}^n_J/\partial\nu \\ d\nu/dt=-\partial\tilde{H}^n_J/\partial\lambda. \end{cases}$$

It is clear that for the equations $\tilde{\mathrm{L}}^n_{\mathrm{VI}}$–$\tilde{\mathrm{L}}^n_{\mathrm{I}}$ we have a theorem similar to Theorem 1 and that for the systems $\tilde{\mathrm{P}}^n_{\mathrm{VI}}$–$\tilde{\mathrm{P}}^n_{\mathrm{I}}$ we have a theorem similar to Theorem 2.

§6. Isomonodromic deformation. First we consider the isomonodromic deformation for the equations $\tilde{\mathrm{L}}^n_J$. Suppose that a, b, c, d do not depend on the deformation parameter t. (For some equations, the independency from t of all or some of a, b, c, d is evident.) The first task is to determine a function A satisfying (2.2). Utilizing the canonical expression of solutions at each singularity of $\tilde{\mathrm{L}}^n_J$ and the relation (2.3), we infer that A is given by

$$A=\begin{cases} Kx+L+\sum_{k=1}^n M_k/(x-\lambda)^k & \text{for } \tilde{\mathrm{L}}^n_{\mathrm{VI}}, \tilde{\mathrm{L}}^n_{\mathrm{V}} \text{ and } \tilde{\mathrm{L}}^n_{\mathrm{III}} \\ L+\sum_{k=1}^n M_k/(x-\lambda)^k & \text{for } \tilde{\mathrm{L}}^n_{\mathrm{IV}} \\ \sum_{k=1}^n M_k/(x-\lambda)^k & \text{for } \tilde{\mathrm{L}}^n_{\mathrm{II}} \text{ and } \tilde{\mathrm{L}}_{\mathrm{I}}, \end{cases}$$

where K, L, M_k are functions in t, λ, μ. Inserting p and A into (2.2), then expanding the left hand side into partial fractions and finally equating the coefficients to zero, we get a system of relations. The equation $\tilde{\mathrm{L}}^n_J$ can be deformed in an isomonodromic way, if the system of relations is compatible. If so, we obtain a system of deformation equations for $\tilde{\mathrm{L}}^n_J$.

It is easy to see that if $\tilde{\mathrm{L}}^n_J$ is deformed isomonodromically, so is L^n_J and that, from a system of deformation equations of $\tilde{\mathrm{L}}^n_J$, the corresponding system of deformation equations of L^n_J is derived at once.

Okamoto showed that the systems of deformation equations of L^1_J and $\tilde{\mathrm{L}}^1_J$ are given by P^1_J and $\tilde{\mathrm{P}}^1_J$ respectively and that the P^1_J ($J=\mathrm{VI}, \cdots \mathrm{I}$) are the Painlevé systems.

Calculations of checking whether $\tilde{\mathrm{L}}^n_J$ can be deformed become enormous rapidly as n increases. We want to make the following conjecture, however:

Conjecture. *For every n and J, P^n_J is a system of deformation equations of L^n_J and $\tilde{\mathrm{P}}^n_J$ is a system of deformation equations of $\tilde{\mathrm{L}}^n_J$.*

We have only the following partial answer to the conjecture.

Theorem 3. *For $n=2,3$ and $J=\mathrm{VI},\cdots,\mathrm{I}$, the conjecture is true.* The conjecture is also true for $n=4,5$ and $J=\mathrm{I}$.

We have from Theorems 1–3 the following commutative diagrams for $n=1,2,3$:

$$\begin{tikzcd}
 & & & L^n_{\mathrm{IV}} \arrow[dr] \arrow[ddd] & & \\
L^n_{\mathrm{VI}} \arrow[r] \arrow[ddd] & L^n_{\mathrm{V}} \arrow[urr] \arrow[dr] \arrow[ddd] & & & L^n_{\mathrm{II}} \arrow[r] \arrow[ddd] & L^n_{\mathrm{I}} \arrow[ddd] \\
 & & L^n_{\mathrm{III}} \arrow[urr] \arrow[ddd] & & & \\
 & & & P^n_{\mathrm{IV}} \arrow[dr] & & \\
P^n_{\mathrm{VI}} \arrow[r] & P^n_{\mathrm{V}} \arrow[urr] \arrow[dr] & & & P^n_{\mathrm{II}} \arrow[r] & P^n_{\mathrm{I}}. \\
 & & P^n_{\mathrm{III}} \arrow[urr] & & &
\end{tikzcd}$$

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