64. *On Symplectic Euler Factors of Genus Two*

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This summarizes the results of our recent attempt to find a "genus-two version" of Eichler's correspondence [1], [2]. Details will be published elsewhere. After [1], [2], several authors have studied the correspondence between automorphic forms belonging to discrete subgroups of $SU(2)$ and of $SL(2, \boldsymbol{R})$ which preserves L functions, notably, [12], [7]. For the groups of higher rank, Ihara [6] studied automorphic forms on $USp(4) \cong \{g \in M_2(\boldsymbol{H})\,;\, g^t\bar{g} = 1_2\}$ ($\boldsymbol{H}$: the Hamilton quaternions), and, as a generalization of Eichler's correspondence, suggested to consider the correspondence between automorphic forms belonging to discrete subgroups of $USp(4)$ and of $Sp(2, \boldsymbol{R})$ (symplectic group of size 4). This problem can be regarded as a special case of the problem of functoriality with respect to L groups proposed later by Langlands (cf. [9], [10]). Let ρ_ν be the representation of $USp(4)$ corresponding to the Young diagram $\begin{array}{|c|c|c|}\hline 1 & \cdots & \nu \\ \hline 1 & \cdots & \nu \\ \hline \end{array}$. Ihara clarified, among others, that the weight of the Siegel modular forms which would correspond to automorphic forms on $USp(4)$ with 'weight ρ_ν' should be $\nu+3$, by showing some character relations between ρ_ν and holomorphic discrete series representations of $Sp(2, \boldsymbol{R})$. But there has been no known example of such a correspondence at all, and we did not know either, which discrete subgroup of $USp(4)$ should correspond to which discrete subgroup of $Sp(2, \boldsymbol{R})$. In this note, we give some examples of pairs of automorphic forms, each of which consists of automorphic forms of $Sp(2, \boldsymbol{R})$ and $USp(4)$ whose Euler 3-factors coincide with each other. This coincidence does not seem accidental, since the coefficients of the Euler factors are fairly large. These Euler 3-factors satisfy the Ramanujan Conjecture, and are obtained from 'new forms' (which can not be obtained as 'liftings' of the forms of one variable, and are not contained in the linear span of automorphic forms belonging to any 'larger' discrete subgroups). We also propose a conjecture which seems reasonable for the present.

§ 1. **Conjecture.** Let D be a definite quaternion algebra over $\boldsymbol{Q}$ with the prime discriminant p, and $\mathcal{O}$ be a maximal order of D. Put $G = \{g \in M_2(D)\,;\, g^t\bar{g} = n(g)1_2,\ n(g) \in \boldsymbol{Q}^\times\}$. In the typical case of Eichler's correspondence, automorphic forms on the adelization $D_A^\times$ of $D^\times$ belonging to $\boldsymbol{H}^\times \prod_{q<\infty} \mathcal{O}_q^\times$ ($\boldsymbol{H}=D\otimes\boldsymbol{R}$) correspond with those belonging to $\Gamma_0(p)\subset SL(2,\boldsymbol{R})$. But in the case of genus two, there are large gaps between the "main terms" (the contribution of the identity element to the dimension of automorphic forms by means of the trace formula) of 'level one' subgroups of G and $\Gamma_0(p)$-type subgroups of $Sp(2,\boldsymbol{Q})$. On the other hand, we know that any reductive algebraic group over a local field has the unique minimal parahoric subgroup up to conjugation. So, it seems natural to consider the correspondence between automorphic forms belonging to (global) discrete subgroups which are obtained from open subgroups of the adelization whose p-components are minimal parahoric. This means that we should consider 'level π' discrete subgroups also for G, where π is a prime element of $\mathcal{O}_p=\mathcal{O}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_p$. More precisely, put

$$G_p^*=\left\{g\in M_2(D_p)\,;\ g\begin{pmatrix}0&1\\1&0\end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix}0&1\\1&0\end{pmatrix},\ n(g)\in Q_p^\times\right\},$$

where $D_p=D\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_p$. For any prime q, let G_q be the q-component of the adelization G_A of G. Then, G_p^* is isomorphic to G_p, and we fix such an isomorphism. Put

$$U_p^*=\begin{pmatrix}\mathcal{O}_p&\mathcal{O}_p\\ \pi\mathcal{O}_p&\mathcal{O}_p\end{pmatrix}^\times\cap G_p^*\quad\text{and}\quad U_q=M_2(\mathcal{O}_q)^\times\cap G_q\qquad\text{for } q\neq p.$$

Put $U_0(D)=G_\infty U_p^*\prod_{q\neq p}U_q\subset G_A$, where G_∞ is the infinite part of G_A. Now, we define the space $\mathfrak{M}_\nu(U_0(D))$ of automorphic forms of G_A with 'weight ρ_ν' belonging to $U_0(D)$. Regard $(x,y)\in\boldsymbol{H}^2$ as the variable over eight dimensional vector space over $\boldsymbol{R}$. Denote by $\mathfrak{M}_\nu$ the $\boldsymbol{R}$ vector space of real valued homogeneous polynomial functions $f(x,y)$ on $\boldsymbol{H}^2$ of degree 2ν which satisfy

(1) $f(ax,ay)=N(a)^\nu f(x,y)$ for any $a\in\boldsymbol{H}$, and

(2) $\Delta f=0$,

where N is the reduced norm of $\boldsymbol{H}$ and Δ is the usual Laplacian with respect to the metric $N(x)+N(y)$ of $\boldsymbol{H}^2$. Then, G acts on $\mathfrak{M}_\nu$ as $f(x,y)\to f((x,y)g)$ for $g\in G$. This representation is an extension of ρ_ν to G, which will be also denoted by ρ_ν. Then, $\mathfrak{M}_\nu(U_0(D))$ is the set of $\mathfrak{M}_\nu$-valued functions f on G_A such that

(1) $f(gh)=f(g)$ for all $h\in G$, and

(2) $f(ug)=\rho_\nu(u_\infty)f(g)$ for all $u\in U_0(D)$,

where u_∞ is the infinite component of u. For an integer n prime to p, put $T(n)=\bigcup_g U_0(D)gU_0(D)$, where g runs through the elements of G_A whose similitudes are n. Put $T(n)=\bigcup_i g_iU_0(D)$ (disjoint). Then, the action of $T(n)$ on $\mathfrak{M}_\nu(U_0(D))$ is defined by:

$$(T(n)f)(g)=\sum_i \rho_\nu(g_i)f(g_i^{-1}g).$$

On the other hand, put

$$B(p)=\left\{g \in Sp(2, Z)\,;\ g \equiv \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \bmod. p\right\},$$

where $*$ runs through integers. Denote by $S_k(B(p))$ the space of Siegel cusp forms with weight k belonging to $B(p)$. The Hecke operator $T(n)(p \nmid n)$ and its action on $S_k(B(p))$ are defined as usual.

Conjecture. *For each even integer $k \geq 4$, there exists a C linear isomorphism i_k of 'new forms' of $\mathfrak{M}_{k-3}(U_0(D))$ to 'new forms' of $S_k(B(p))$ such that $L(s, i_k(f))=L(s, f)$ up to Euler p-factors for any common eigen 'new form' f of $\mathfrak{M}_{k-3}(U_0(D))$ of all the Hecke operators $T(n)(p \nmid n)$.*

Here, we define new forms of $\mathfrak{M}_\nu(U_0(D))$ (resp. $S_k(B(p))$) to be the elements of the orthogonal complement of the space spanned by automorphic forms of G_A (resp. cusp forms of $Sp(2, \boldsymbol{Q}_A)$) belonging to any larger subgroups of G_A (resp. $Sp(2, \boldsymbol{Q}_A)$) containing $U_0(D)$ (resp. $Sp(2, \boldsymbol{R}) \prod_{q \neq p} Sp(2, Z_q) B(p)_p$, where $B(p)_p$ is the topological closure of $B(p)$ in $Sp(2, \boldsymbol{Q}_p)$). We denote by $L(S, *)$ the (denominator of the) L function of Andrianov type.

§2. Examples. Put $D=\boldsymbol{Q}+\boldsymbol{Q}i+\boldsymbol{Q}j+\boldsymbol{Q}k$, $i^2=-1$, $j^2=-1$, $ij=-ji=k$, and $\mathcal{O}=\boldsymbol{Z}+\boldsymbol{Z}i+\boldsymbol{Z}j+\boldsymbol{Z}(1+i+j+k)/2$. Then, the discriminant of D is two and $\mathcal{O}$ is a maximal order of D. We can show that the 'class number' of $U_0(D)$ (that is, the number of the double cosets in $G \backslash G_A / U_0(D)$) is one. Then, $\mathfrak{M}_\nu(U_0(D))$ can be identified with

$$\mathfrak{M}_\nu(\Gamma_0)=\{f \in \mathfrak{M}_\nu\,;\ f((x, y)\gamma)=f(x, y) \text{ for all } \gamma \in \Gamma_0\},$$

where

$$\Gamma_0=\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix}; a, d \in \mathcal{O}^\times, N(a-d) \equiv 0 \bmod. 2\right\}.$$

Under this identification, the Hecke operator $T(n)(2 \nmid n)$ acts on $\mathfrak{M}_\nu(\Gamma_0)$ as

$$f(x, y) \longrightarrow (T(n)f)(x, y)=\sum_{g \in \Delta_n / \Gamma_0} f((x, y)g),$$

where

$$\Delta_n=\left\{g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \cap M_2(\mathcal{O})\,;\ n(g)=n \text{ and } N(a-d) \equiv N(b-c) \equiv 0 \bmod. 2\right\}.$$

On the other hand, by using Igusa [5], the graded ring $A(B(2))$ of modular forms belonging to $B(2)$ and the ideal of cusp forms in $A(B(2))$ can be given explicitly in terms of theta constants, together with an explicit dimension formula. Now, let $f \in \mathfrak{M}_\nu(U_0(D))$ or $S_k(B(2))$ be a common eigen form of all $T(n)(2 \nmid n)$. Then, the Hecke polynomial of f at a prime $q \neq 2$ is defined by

$$H_q(T, f)=T^4-\lambda(q)T^3+(\lambda(q)^2-\lambda(q^2)-q^{2m-4})T^2-\lambda(q)q^{2m-3}T+q^{4m-6},$$

where $m=k$ or $\nu+3$ for $f\in S_k(B(2))$ or $\mathfrak{M}_\nu(U_0(D))$, respectively, and $\lambda(q)$ or $\lambda(q^2)$ is the eigen value of $T(q)$ or $T(q^2)$ on f, respectively. Then, $H_q(q^s, f)q^{-4s}$ is the Euler q-factor of $L(s, f)$. Denote by $\mathfrak{M}_\nu^0(\Gamma_0)$ or $S_k^0(B(2))$ the space of new forms of $\mathfrak{M}_\nu(\Gamma_0)$ or $S_k(B(2))$, respectively. For small odd ν and even k, we obtain the following table:

ν	1	3	5	7	9
$\dim \mathfrak{M}_\nu(\Gamma_0)$	0	0	1	1	2
$\dim \mathfrak{M}_\nu^0(\Gamma_0)$	0	0	0	1	1

k	2	4	6	8	10	12
$\dim S_k(B(2))$	0	0	1	3	6	12
$\dim S_k^0(B(2))$	0	0	0	0	1	1

Define the real valued functions $x_i=x_i(x)$ $(i=1,\cdots,4)$ on $\boldsymbol{H}$, by $x=x_1+x_2i+x_3j+x_4k$. Put

$$f_7(x,y)=(N(y)-N(x))(N(x)^2-3N(x)N(y)+N(y)^2)\prod_{i=1}^{4}(\bar{y}x)_i,$$

and

$$f_9(x,y)=(N(y)-N(x))(153N(x)^4-1122N(x)^3N(y)+2618N(x)^2N(y)^2$$
$$-1122N(x)N(y)^3+153N(y)^4-1292\sum_{i=1}^{4}(\bar{y}x)_i^4)\prod_{i=1}^{4}(\bar{y}x)_i^4.$$

Put also $X=(\theta_{0000}^4+\theta_{0001}^4+\theta_{0010}^4+\theta_{0011}^4)/4$, $Y=(\theta_{0000}\theta_{0010}\theta_{0001}\theta_{0011})^2$, $Z=(\theta_{0100}^4-\theta_{0110}^4)^2/16384$, $T=(\theta_{0100}\theta_{0110})^4/256$, $R=(X^2-Y-1024Z-64T)/64$, and $K=(\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096$, where $\theta_m(\tau)$ is a theta constant on the Siegel upper half space of genus two given by

$$\theta_m(\tau)=\sum_{p\in \boldsymbol{Z}^2}\exp 2\pi i[{}^t(p+m'/2)\tau(p+m'/2)/2+{}^t(p+m'/2)m''/2]$$

for any $m={}^t(m', m'')$, $m', m''\in \boldsymbol{Z}^2$. Put

$$F_{10}=12XTR-2XYZ+X^2K+YK+1024ZK+96RK,$$

and

$$F_{12}=36YTR+36864ZTR+3840TR^2-1920RYZ+12X^2TR$$
$$-21Y^2Z-21504YZ^2+XYK+1024XZK-3840K^2$$
$$+13X^2YZ+7X^3K.$$

Theorem. *Bases of $\mathfrak{M}_\nu^0(\Gamma_0)$ or $S_k^0(B(2))$ for $\nu=7, 9$, and $k=10, 12$, are given respectively as follows:*

$$\mathfrak{M}_7^0(\Gamma_0)=\boldsymbol{C}f_7(x,y),\qquad \mathfrak{M}_9^0(\Gamma_0)=\boldsymbol{C}f_9(x,y),$$
$$S_{10}^0(B(2))=\boldsymbol{C}F_{10},\qquad S_{12}^0(B(2))=\boldsymbol{C}F_{12}.$$

The Hecke polynomials of these automorphic forms at $q=3$ are given by:

$$H_3(T,f_7)=H_3(T,F_{10})=T^4+18360T^3+297016470T^2+3^{17}\cdot 18360T+3^{34}$$
$$=(T^2+108(85-8\sqrt{61})T+3^{17})(T^2+108(85+8\sqrt{61})T+3^{17}),$$

and

$$H_3(T,f_9)=H_3(T,F_{12})=T^4+14760T^3-9330332490T^2+3^{21}\cdot 14760T+3^{42}$$
$$=(T^2+36(205+2\sqrt{5845969})T+3^{21})$$
$$\times(T^2+36(205-2\sqrt{5845969})T+3^{21}).$$

The absolute values of the zeros of these polynomials are equal to $3^{17/2}$ and $3^{21/2}$, respectively.

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