60. *Further Results on the Boundedness and the Attractivity Properties of Nonlinear Second Order Differential Equations*

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1. Introduction. Recently in [2], J. R. Graef and P. W. Spikes discussed the boundedness of solutions of the forced second order nonlinear nonautonomous differential equation

$$(1)\qquad (a(t)x')'+h(t,x,x')+q(t)f(x)g(x')=e(t,x,x').$$

In [4], we discussed the boundedness of solutions of (1) and the attractivity properties of the equation

$$(2)\qquad (a(t)x')'+p(t)f_1(x)g_1(x')x'+q(t)f_2(x)g_2(x')x=e(t,x,x')$$

and obtained the results which are strict extensions of ones in [2] and in [1]. The purpose of this paper is to give the proofs of Remarks 2–4 in [4].

2. Theorems and proofs. First, we consider the boundedness of solutions of the equation (1) or an equivalent system of equations

$$(3)\qquad \begin{aligned} x'&=y\\ y'&=\frac{1}{a(t)}\{-a'(t)y-h(t,x,y)-q(t)f(x)g(y)+e(t,x,y)\}\end{aligned}$$

under the following assumptions.

(A_1) *$a(t)$ and $q(t)$ are positive C^1-functions in $I=[0,\infty)$.*

(A_2) *$f(x)$ is a continuous function in R^1 which satisfies*

$$\int_0^{\pm\infty} f(x)dx=\infty.$$

(A_3) *$g(y)$ is a continuous, positive function in R^1.*

(A_4) *$h(t,x,y)$ is a continuous function in $I\times R^2$ which satisfies the inequality $yh(t,x,y)\geqq 0$.*

(A_5) *$e(t,x,y)$ is a continuous function in $I\times R^2$.*

In what follows, we shall use the notations $a'(t)_+=\max\{a'(t),0\}$ and $a'(t)_-=\max\{-a'(t),0\}$. We shall also use

$$F(x)=\int_0^x f(u)du \quad\text{and}\quad G(y)=\int_0^y \frac{v}{g(v)}dv.$$

Theorem 1. *Suppose* (A_1)–(A_5) *and the following conditions.*

$$(4)\qquad \int_0^\infty \frac{|a'(t)|}{a(t)}dt<\infty,\qquad \int_0^\infty \frac{q'(t)_-}{q(t)}dt<\infty.$$

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(5) $\dfrac{y^2}{g(y)} \leqq MG(y)$ *in* $|y| \geqq k$ *for some* $M>0$ *and* $k \geqq 0$.

(6) *There exist continuous, nonnegative functions* $r_1(t)$ *and* $r_2(t)$ *satisfying*

$$|e(t,x,y)| \leqq \frac{a(t)\,|q'(t)|}{Mq(t)} + r_1(t) + r_2(t)\,|y|, \qquad \int_0^\infty r_i(t)dt < \infty \quad (i=1,2).$$

Then any solution $x(t)$ *of* (1) *is bounded.*

If, in addition, the functions $G(y)$ *and* $q(t)$ *satisfy the condition:*

(7) $G(y) \to \infty$ *as* $|y| \to \infty$, $q(t) \leqq q_2$ *for some constant* q_2,

then any solution $(x(t), y(t))$ *of* (3) *is bounded.*

Remark 1. It follows from (4) that there exist positive constants a_1, a_2 and q_1 which satisfy $a_1 \leqq a(t) \leqq a_2$ and $q_1 \leqq q(t)$ in I. The assumption (A_3) and the condition (5) imply that there exist constants $M'>0$ and $m \geqq 0$ such that

$$\frac{y^2}{g(y)} \leqq M'G(y), \qquad \frac{|y|}{g(y)} \leqq m + MG(y) \quad \text{in } R^1.$$

Proof of Theorem 1. Since (A_2) implies that $F(x) \to \infty$ as $|x| \to \infty$, there exists a positive number F_0 satisfying the inequality $F(x)+F_0 \geqq 0$ for arbitrary x in R^1. Let

$$V_1(t,x,y) = \left[\frac{q(t)}{a(t)}(F(x)+F_0) + G(y) + \frac{m}{M}\right] \times \exp\left\{-\int_0^t \frac{a'(s)_-}{a(s)}ds + 2\int_0^t \frac{q'(s)_-}{q(s)}ds\right\}.$$

Differentiating $V_1(t) \equiv V_1(t, x(t), y(t))$ with respect to t for any solution $(x(t), y(t))$ of (3), then we have

$$V_1'(t) \leqq \left\{\frac{|q'(t)|}{q(t)} + 2\frac{q'(t)_-}{q(t)} + M'\frac{a'(t)_-}{a(t)} + M\frac{r_1(t)}{a(t)} + M'\frac{r_2(t)}{a(t)}\right\}V_1(t) + \left\{\frac{2mq'(t)_-}{Mq(t)} + m\frac{r_1(t)}{a(t)}\right\}\exp\left\{2\int_0^t \frac{q'(s)_-}{q(s)}ds\right\} \quad \text{for any } t \geqq 0.$$

Integrating the above inequality from t_0 to t, and using Gronwall's lemma, we obtain from (4) and (6) that

$$\begin{aligned} V_1(t) \leqq & \left[V_1(t_0) + \int_0^\infty \left\{\frac{2mq'(s)_-}{Mq(s)} + \frac{m}{a_1}r_1(s)\right\}ds \cdot \exp\left\{2\int_0^\infty \frac{q'(s)_-}{q(s)}ds\right\}\right] \\ & \times \exp\left[\int_{t_0}^t \frac{q'(s)}{q(s)}ds + \int_0^\infty \left\{4\frac{q'(s)_-}{q(s)} + M'\frac{a'(s)_-}{a(s)} + \frac{M}{a_1}r_1(s) + \frac{M'}{a_1}r_2(s)\right\}ds\right] \\ \leqq & \, c_2 q(t) \qquad \text{for } t \geqq t_0. \end{aligned}$$

Now it follows that for $t \geqq t_0$,

$$F(x(t)) \leqq c_2 a_2 \exp\left\{\int_0^\infty \frac{a'(s)_-}{a(s)}ds\right\}$$

and

$$G(y(t)) \leqq c_2 q(t) \exp\left\{\int_0^\infty \frac{a'(s)_-}{a(s)} ds\right\}.$$

The proof of Theorem 1 is now completed by (A_2) and (7). Q.E.D.

Corollary 1. *Suppose* (A_1)–(A_5), (6) *and the following conditions*:

(8) $a'(t) \geqq 0$, $a(t) \leqq a_2$ *for a constant* $a_2 > 0$ *and* $\int_0^\infty \frac{q'(t)_-}{q(t)} dt < \infty$.

(9) *There exist constants* $M > 0$ *and* $m \geqq 0$ *such that*

$$\frac{|y|}{g(y)} \leqq m + MG(y) \qquad \text{in } R^1.$$

Then any solution $x(t)$ *of* (1) *is bounded.*

If, in addition, the condition (7) *holds, then any solution* $(x(t), y(t))$ *of* (3) *is bounded.*

The proof of Corollary 1 is similar to that of Theorem 1 and we shall omit its details.

Next, we consider the attractivity properties of the equation (2) or an equivalent system

$$(10) \qquad \begin{aligned} x' &= y \\ y' &= \frac{1}{a(t)}\{-a'(t)y - p(t)f_1(x)g_1(y)y - q(t)f_2(x)g_2(y)x + e(t, x, y)\} \end{aligned}$$

under the assumptions (A_1), (A_5) and the following assumptions.

(A_6) $\int_0^\infty \frac{|a'(t)|}{a(t)} dt < \infty, \qquad \int_0^\infty \frac{|q'(t)|}{q(t)} dt < \infty.$

(A_7) $p(t)$ *is a continuous function in* I *satisfying* $p_1 \leqq p(t) \leqq p_2$ *for some positive constants* p_1 *and* p_2.

(A_8) $f_1(x)$ *and* $f_2(x)$ *are continuous, positive functions in* R^1 *and* $f_2(x)$ *satisfies* $\int_0^{\pm\infty} x f_2(x) dx = +\infty$.

(A_9) $g_1(y)$ *and* $g_2(y)$ *are continuous, positive functions in* R^1 *and* $g_2(y)$ *satisfies* $\int_0^{\pm\infty} \frac{y}{g_2(y)} dy = +\infty$.

Remark 2. If we assume $\int_0^\infty \frac{q'(t)_-}{q(t)} dt < \infty$, then the latter of (A_6) follows from the condition $q(t) \leqq q_2$ for $t \in I$.

On the other hand, (A_6) implies the existence of positive constants a_1, a_2, q_1 and q_2 such that $a_1 \leqq a(t) \leqq a_2$ and $q_1 \leqq q(t) \leqq q_2$ for $t \in I$.

From now on, we shall use the following functions:

$$F_1(x) = \int_0^x f_1(u) du, \quad F_2(x) = \int_0^x u f_2(u) du, \quad G_0(y) = \int_0^y \frac{v}{g_2(v)} dv,$$

$$G_1(y) = \int_0^y \frac{1}{g_1(v)} dv \quad \text{and} \quad G_2(y) = LG_0(y) - \frac{1}{2}\{G_1(y)\}^2$$

where L is a positive constant to be determined later.

Theorem 2. *Suppose* (A_1), (A_5)–(A_9), (6) *and the following condition.*

(11) *There exist constants* $M > 0$ *and* $k \geqq 0$ *such that*

$$\frac{y^2}{g_2(y)} \leqq MG_0(y) \qquad in\ |y| \geqq k.$$

Then every solution of (10) *approaches* (0, 0) *as* $t \to \infty$.

Proof of Theorem 2. The boundedness of solutions of (10) is an immediate consequence of Theorem 1, since (A_8) implies $F_2(x) \to +\infty$ as $|x| \to \infty$ and since (A_9) implies $G_0(y) \to +\infty$ as $|y| \to \infty$. Let $(x(t), y(t))$ be a solution defined in $[t_0, \infty)$ of (10), then there exists a constant K such that $|x(t)|+|y(t)| \leqq K$ for $t \geqq t_0$. It follows from (A_8) and (A_9) that there exist positive constants $c_1, c_2, \cdots, c_8$ such that

(12) $c_1 \leqq f_1(x) \leqq c_2$, $c_3 \leqq f_2(x) \leqq c_4$, $c_5 \leqq g_1(y) \leqq c_6$ and $c_7 \leqq g_2(y) \leqq c_8$ in $|x|+|y| \leqq K$. Let

$$V_2(t, x, y) = \frac{1}{2q(t)}(F_1(x)+G_1(y))^2 + \frac{L}{a(t)} F_2(x) + \frac{1}{q(t)} G_2(y)$$

for $(t, x, y) \in I \times R^2$, then we have for $t \in I$, $|x|+|y| \leqq K$

$$V_2(t, x, y) \geqq \frac{L}{a(t)} F_2(x) + \frac{1}{q(t)} G_2(y) \geqq \frac{c_3 L}{2a_2} x^2 + \frac{1}{2q_2}\left(\frac{L}{c_8} - \frac{1}{c_5^2}\right) y^2$$

and

$$V_2(t, x, y) = \frac{1}{2q(t)}\{F_1(x)^2 + 2F_1(x)G_1(y)\} + \frac{L}{a(t)} F_2(x) + \frac{1}{q(t)} G_0(y)$$
$$\leqq \left(\frac{c_2^2}{2q_1} + \frac{c_2}{2q_1 c_5} + \frac{c_4 L}{2a_1}\right) x^2 + \left(\frac{c_2}{2q_1 c_5} + \frac{L}{2q_1 c_7}\right) y^2.$$

Differentiating $V_2(t) = V_2(t, x(t), y(t))$ with respect to t, we obtain

$$V_2'(t) \leqq \frac{|q'(t)|}{q(t)}\left\{\frac{1}{2q(t)}(F_1(x)+G_1(y))^2 + \frac{1}{q(t)}|G_2(y)|\right\} + \frac{|a'(t)|}{a(t)}\left\{\frac{|yF_1(x)|}{q(t)g_1(y)}\right.$$
$$\left. + \frac{L}{a(t)} F_2(x) + \frac{Ly^2}{q(t)g_2(y)}\right\} + \left(\frac{|q'(t)|}{Mq(t)^2} + \frac{r_1(t)}{a(t)q(t)}\right) \cdot \left(\frac{|F_1(x)|}{g_1(y)}\right.$$
$$\left. + \frac{L|y|}{g_2(y)}\right) + \frac{r_2(t)}{a(t)q(t)}\left(\frac{|yF_1(x)|}{g_1(y)} + \frac{Ly^2}{g_2(y)}\right) + \frac{f_1(x)}{q(t)}\left(1 + \frac{p(t)}{a(t)}\right)|yF_1(x)|$$
$$+ \frac{f_1(x)}{q(t)} y G_1(y) - \frac{f_2(x) g_2(y)}{a(t) g_1(y)} x F_1(x) - \frac{Lp(t) f_1(x) g_1(y)}{a(t)q(t)g_2(y)} y^2.$$

We can choose L so large that $(L/c_8 - 1/c_5^2)/2 \geqq 1 + 1/c_7$. Then we get $G_2(y) \geqq (L/c_8 - 1/c_5^2) y^2/2 \geqq y^2$, $y^2/g_2(y) \leqq (1/c_7) y^2 \leqq G_2(y)$ and $c_9(x^2+y^2) \leqq V_2(t, x, y) \leqq c_{10}(x^2+y^2)$ for $t \in I$, $|x|+|y| \leqq K$, where c_9 and c_{10} are positive constants. It is clear that $|yF_1(x)| \leqq c_2|xy| \leqq (c_2/2)(x^2+y^2)$, $yG_1(y) \leqq (1/c_5) y^2$, $xF_1(x) \geqq c_1 x^2$, $|yF_1(x)|/g_1(y) + Ly^2/g_2(y) \leqq (c_2/2c_5)(x^2+y^2) + (L/c_7) y^2 \leqq (c_2/2c_5 + L/c_7)(x^2+y^2)$ and $|F_1(x)|/g_1(y) + L|y|/g_2(y) \leqq (c_2/c_5 + L/c_7)K$ in $|x|+|y| \leqq K$. Analogously, we can show that $f_1(x)/q(t)(1+p(t)/a(t))|yF_1(x)| + (f_1(x)/q(t)) yG_1(y) - (f_2(x) g_2(y)/a(t) g_1(y)) xF_1(x) - (Lp(t) f_1(x) g_1(y)/a(t)q(t)g_2(y)) y^2 \leqq -c_{11}(x^2+y^2)$ in $|x|+|y| \leqq K$ for L large enough, where c_{11} is some positive constant. Thus we have that

$$V_2'(t) \leqq \left[-\frac{c_{11}}{c_{10}} + L_1\left\{\frac{|q'(t)|}{q(t)} + \frac{|a'(t)|}{a(t)} + r_2(t)\right\}\right] V_2(t) + L_2\left\{\frac{|q'(t)|}{q(t)} + r_1(t)\right\}$$

for some $L_1 > 0$ and $L_2 > 0$.

Now, let

$$W(t,x,y)=V_2(t,x,y)\cdot\exp\left[-L_1\int_0^t\left\{\frac{|q'(s)|}{q(s)}+\frac{|a'(s)|}{a(s)}+r_2(s)\right\}ds\right],$$

then we obtain

$$c_9\exp\left[-L_1\int_0^\infty\left\{\frac{|q'(s)|}{q(s)}+\frac{|a'(s)|}{a(s)}+r_2(s)\right\}ds\right](x^2+y^2)\leqq W(t,x,y)$$

for $t\in I$, $|x|+|y|\leqq K$ and also

$$W'(t)\leqq -\frac{c_{11}}{c_{10}}W(t)+L_2\left\{\frac{|q'(t)|}{q(t)}+r_1(t)\right\},$$

where $W(t)=W(t,x(t),y(t))$. The following Lemma completes the proof of Theorem 2. Q.E.D.

Lemma 1. *Consider a system of differential equations*

(S) $x'=f(t,x)$, *where* $f(t,x)$ *is continuous in* $I\times D$, $D=\{x\in R^2\,|\,\|x\|\leqq H\}$, $H>0$ *and* $\|\cdot\|$ *is the Euclidean norm. If there exists a Liapunov function* $U(t,x)$ *defined in* $I\times D$ *such that*

(i) $U(t,x)$ *is continuously differentiable in* $I\times D$,

(ii) $c\|x\|^2\leqq U(t,x)$, *where* c *is a positive constant,*

(iii) $U'_{(s)}(t,x)\leqq -\lambda U(t,x)+r(t)$, *where* λ *is a positive constant and* $r(t)$ *is a continuous, nonnegative function satisfying* $\int_0^\infty r(t)dt<\infty$,

then every solution of (S) *which defined in the future in* D, *approaches the origin as* $t\to\infty$.

The proof is given by the variation of constant formula.

Theorem 3. *Suppose* (A_1), (A_5)–(A_9), (11) *and the following*:

(13) $f_2(x)$ *and* $g_2(y)$ *have positive lower bounds, that is*

$$f_2(x)\geqq\varepsilon>0 \text{ in } R^1 \text{ and } g_2(y)\geqq\delta>0 \text{ in } R^1.$$

(14) *There exist continuous, nonnegative functions* $r_1(t), r_2(t)$ *such that*

$$|e(t,x,y)|\leqq\frac{a(t)\,|q'(t)|}{Mq(t)}+r_1(t)+r_2(t)\{|x|+|y|\},\quad \int_0^\infty r_i(t)dt<\infty\quad (i=1,2).$$

Then every solution of (10) *approaches* $(0,0)$ *as* $t\to\infty$.

Proof of Theorem 3. To show the boundedness of solutions, let

$$V_3(t,x,y)=\left\{\frac{q(t)}{a(t)}F_2(x)+G_0(y)+1\right\}\exp\left[-\int_0^t\left\{\frac{|a'(s)|}{a(s)}+\frac{|q'(s)|}{q(s)}\right\}ds\right].$$

Then we have

$$\begin{aligned}V_3'(t)\leqq&\left[M'\left\{\frac{|a'(t)|}{a(t)}+\frac{r_2(t)}{a(t)}\right\}G_0(y)+\sqrt{\frac{M'}{\delta}}\left\{\frac{|q'(t)|}{Mq(t)}+\frac{r_1(t)}{a(t)}\right\}\sqrt{G_0(y)}\right.\\&\left.+\sqrt{\frac{2M'}{\varepsilon\delta}}\cdot\frac{r_2(t)}{a(t)}\cdot\sqrt{F_2(x)G_0(y)}\right]\exp\left[-\int_0^t\left\{\frac{|a'(s)|}{a(s)}+\frac{|q'(s)|}{q(s)}\right\}ds\right]\\\leqq&L_1\left\{\frac{|a'(t)|}{a(t)}+\frac{|q'(t)|}{q(t)}+r_1(t)+r_2(t)\right\}V_3(t)\qquad\text{for some } L_1>0.\end{aligned}$$

The above estimates are valid, since (11) and (13) imply that

$$\frac{y^2}{g_2(y)} \leqq M'G_0(y), \quad \frac{|y|}{g_2(y)} \leqq \sqrt{\frac{M'}{\delta}G_0(y)} \leqq \frac{1}{2}\sqrt{\frac{M'}{\delta}} \cdot (G_0(y)+1),$$

$$|x| \leqq \sqrt{\frac{2}{\varepsilon}F_2(x)} \quad \text{and} \quad \sqrt{\frac{q(t)}{a(t)}F_2(x)\cdot G_0(y)} \leqq \frac{1}{2}\left\{\frac{q(t)}{a(t)}F_2(x)+G_0(y)\right\}.$$

By Gronwall's lemma, we obtain

$$V_3(t) \leqq V_3(t_0)\exp\left[L_1\int_0^\infty \left\{\frac{|a'(s)|}{a(s)}+\frac{|q'(s)|}{q(s)}r_1(s)+r_2(s)\right\}ds\right]=L_2 \qquad \text{for } t \geqq t_0 \geqq 0.$$

This implies that

$$F_2(x(t)) \leqq \frac{a_2L_2}{q_1}\exp\left[\int_0^\infty \left\{\frac{|a'(s)|}{a(s)}+\frac{|q'(s)|}{q(s)}\right\}ds\right]$$

and

$$G_0(y(t)) \leqq L_2\exp\left[\int_0^\infty \left\{\frac{|a'(s)|}{a(s)}+\frac{|q'(s)|}{q(s)}\right\}ds\right] \qquad \text{for } t \geqq t_0 \geqq 0.$$

Therefore we conclude from (A_8) and (A_9) that every solution of (10) is bounded.

Next, let $(x(t), y(t))$ be a solution defined in $[t_0, \infty)$ of (10) which satisfies $|x(t)|+|y(t)| \leqq K$ in $[t_0, \infty)$ for some $K>0$. We use the same function $V_2(t, x, y)$ as that in the proof of Theorem 2. Then we have

$$V_2' \leqq \left[-\frac{c_{11}}{c_{10}}+L_3\left\{\frac{|q'(t)|}{q(t)}+\frac{|a'(t)|}{a(t)}+r_2(t)\right\}\right]V_2(t)+L_2\left\{\frac{|q'(t)|}{q(t)}+r_1(t)\right\}$$

where L_1, L_2, L_3 are some positive constants. We can get the conclusion of Theorem 3 along the analogous way as the proof of Theorem 2. Q.E.D.

Corollary 2. *Suppose* (A_1), (A_5)–(A_9), (14) *and the following*:

(15) $\frac{|y|}{g_2(y)} \leqq M\sqrt{G_0(y)}, \quad g_2(y) \leqq \gamma \quad$ *in* R^1.

(16) $f_2(x) \geqq \varepsilon > 0 \quad$ *in* R^1.

Then every solution of (10) *approaches* $(0, 0)$ *as* $t \to \infty$.

References

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