59. On the L^2 Boundedness of Fourier Integral Operators in R^n

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§ 1. **Introduction.** A Fourier integral operator A is an operator of the form

$$Af(x)=(2\pi)^{-n}\int_{R^n} e^{iS(x,\xi)}a(x,\xi)\hat{f}(\xi)d\xi, \tag{1.1}$$

where

$$\hat{f}(\xi)=\int_{R^n} e^{-iy\cdot\xi}f(y)dy$$

is the Fourier transform of f defined on R^n. We call $S(x,\xi)$ its phase function and $a(x,\xi)$ its symbol function (cf. Eskin [6], Hörmander [10]).

When $S(x,\xi)=x\cdot\xi$, a Fourier integral operator reduces to a pseudodifferential operator. Beals-Fefferman [2], [3] proved the L^2 boundedness theorem for a quite wide class of pseudodifferential operators. In this note we shall prove the L^2 boundedness theorem of the operator A with general phase function S and with symbol function a in the Beals-Fefferman class. This theorem contains the above-mentioned theorem of Beals-Fefferman as a special case, and the L^2 boundedness theorem of Fourier integral operators in Fujiwara [8] and Kumano-go [12] as well.

§ 2. **Statements and results.** **Definition 2.1** (Beals [3], Hörmander [11]). Let Φ,φ be a pair of positive functions defined on $R^n\times R^n$. We call Φ,φ a pair of weight functions if Φ,φ satisfy

$$\begin{cases} \text{(i)} & \Phi\geqq c_1,\quad \varphi\leqq C_2,\quad \Phi\varphi\geqq c_3; \\ \text{(ii)} & \Phi(x,\xi)\approx\Phi(y,\eta),\quad \varphi(x,\xi)\approx\varphi(y,\eta) \\ & \text{whenever } |x-y|\leqq r_0\varphi(y,\eta),\quad |\xi-\eta|\leqq r_0\Phi(y,\eta) \\ & (A\approx B \text{ means that } C^{-1}\leqq A/B\leqq C \text{ for some positive} \\ & \quad\text{constant } C); \\ \text{(iii)} & \dfrac{\Phi(x,\xi)}{\Phi(y,\eta)}+\dfrac{\varphi(x,\xi)}{\varphi(y,\eta)}\leqq C_4(1+\Phi(y,\eta)\,|x-y|+\varphi(y,\eta)\,|\xi-\eta|)^N; \end{cases} \tag{2.1}$$

for some positive constants c_1, C_2, c_3, C_4, r_0 and a non-negative constant N.

Let a pair of weight functions Φ,φ be fixed. Our assumptions are:

(A-1) $a(x,\xi)$ is in $S^{0,0}_{\Phi,\varphi}$, that is, for any integer $k\geqq 0$

$$|a|_k=\max_{|\alpha+\beta|\leqq k}\ \sup_{(x,\xi)\in R^n\times R^n} |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\,\varphi(x,\xi)^{|\alpha|}\Phi(x,\xi)^{|\beta|}$$

is finite.

(A-2) The real part S_R of S satisfies the estimate

$$\inf_{(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{R}^n} |\det [\partial_{x_j}\partial_{\xi_k} S_R(x,\xi)]| \geqq \delta_0$$

for some positive constant δ_0.

(A-3) For any pair of multi-indices α, β for $|\alpha+\beta|\geqq 2$, the inequality

$$|\partial_x^\alpha \partial_\xi^\beta S(x,\xi)| \leqq C_{\alpha,\beta}\varphi(x,\xi)^{1-|\alpha|}\Phi(x,\xi)^{1-|\beta|}$$

holds.

(A-4) The imaginary part $S_I(x,\xi)$ of $S(x,\xi)$ is non-negative.

Then we see from (A-1) and (A-4) that the defining integral in (1.1) is absolutely convergent at least for f in $\mathcal{S}(\boldsymbol{R}^n)$.

We denote the norm in $L^2(\boldsymbol{R}^n)$ by $\| \quad \|$. Our result is:

Theorem. *Let $a(x,\xi)$ and $S(x,\xi)$ be two C^∞ functions satisfying the assumptions* (A-1)–(A-4). *Then there exists a constant $C>0$ such that for any f in $\mathcal{S}(\boldsymbol{R}^n)$ we have*

$$\|Af\| \leqq C\|f\|. \tag{2.2}$$

Remarks. 1. When $\Phi=(1+|\xi|)^\rho$, $\varphi=(1+|\xi|)^{-\delta}$, $0\leqq\delta\leqq\rho\leqq 1$, $\delta<1$, the assumption (A-1) is that $a(x,\xi)$ is in $S^0_{\rho,\delta}$ in the notation of Hörmander [10]. This theorem contains the result of Fujiwara [8] and Kumano-go [12].

2. When $\Phi=\varphi=1$, the operator A turns out to be the oscillatory integral transformation in Fujiwara [7], Asada-Fujiwara [1]. Fujiwara [9] used these operators to construct the fundamental solutions of Schrödinger's operator.

3. Danilov [5] considered the operator A under the assumption that $e^{-S_I(x,\xi)}a(x,\xi)$ is in $S^0_{\rho,\rho}$ instead of (A-1).

§ 3. Outline of the proof of Theorem. By the Plancherel's theorem we have only to prove that the integral operator

$$u(\xi) \longmapsto \int_{\boldsymbol{R}^n} e^{iS(x,\xi)} a(x,\xi) u(\xi) d\xi \tag{3.1}$$

is L^2 bounded. We still denote by A this integral operator.

To prove L^2 boundedness we shall use the following partition of unity. Choose a non-increasing C^∞ function ψ on $\boldsymbol{R}^1$ so that $\psi(t)=1$ for $t<R'$, $\psi(t)=0$ for $t>R$, for some R, R' with $0<R'<R<(1/4)r_0$. And for any $(s,\sigma)\in \boldsymbol{R}^n\times\boldsymbol{R}^n$ set

$$\psi_{(s,\sigma)}(x,\xi)=\frac{\psi(\lambda(s,\sigma)\,|x-s|)\psi(\lambda(s,\sigma)^{-1}\,|\xi-\sigma|)}{\iint_{\boldsymbol{R}^{2n}} \psi(\lambda(s,\sigma)\,|x-s|)\psi(\lambda(s,\sigma)^{-1}\,|\xi-\sigma|)\,ds d\sigma},$$

where we put

$$\lambda(s,\sigma)=\sqrt{\Phi(s,\sigma)/\varphi(s,\sigma)}.$$

Then we can prove the following lemma (cf. Hörmander [11]).

Lemma 3.1. 1) *Each $\psi_{(s,\sigma)}$ is supported in the set*

$$U_{(s,\sigma)}(R)=\{(x,\xi)\,;\,|x-s|\leqq R\lambda(s,\sigma)^{-1},\ |\xi-\sigma|\leqq R\lambda(s,\sigma)\}.$$

2) *For any positive integer* m $|\psi_{(s,\sigma)}|_m\leqq C_m$.

3) $\iint_{R^{2n}}\psi_{(s,\sigma)}(x,\xi)dsd\sigma=1$ *for any* (x,ξ) *in* $R^n\times R^n$.

For $p=(s,\sigma)$ in $R^n\times R^n$ set $a_p(x,\xi)=a(x,\xi)\psi_p(x,\xi)$ and define

$$A_pu(x)=\int_{R^n}e^{iS(x,\xi)}a_p(x,\xi)u(\xi)d\xi. \tag{3.2}$$

Then we can prove the following proposition as in [1, Lemma 2.1].

Proposition 3.2. *Let* $u(\xi)$ *be in* $C_0^\infty(R^n)$. *Then*

1) $A_pu(x)\in C_0^\infty(R^n)$.

2) $\|A_pu(x)\|\leqq C\|u\|$, *where the constant* C *is independent of* $p=(s,\sigma)$.

3) $Au(x)=\lim_{j\to\infty}\iint_{|s|+|\sigma|\leqq j}A_{(s,\sigma)}u(x)dsd\sigma$,

where the limit exists at every x *and with respect to the strong topology in* $L^2(R^n)$ *as well.*

Therefore it is sufficient for the proof of Theorem to prove the following

Proposition 3.3. *For any compact set* K *in* $R^n\times R^n$ *we have the estimate*

$$\left\|\int_K A_pu(x)dp\right\|\leqq M\|u\|,\qquad u\in C_0^\infty(R^n), \tag{3.3}$$

where the constant M *is independent of* K *and* u.

To prove Proposition 3.3 we shall appeal to the following lemma. See Calderón-Vaillancourt [4].

Lemma 3.4. *Let* $h(p,p')$ *and* $k(p,p')$ *be two positive functions on* $R^{2n}\times R^{2n}$ *such that*

$$\|A_pA_{p'}^*\|\leqq h(p,p')^2,\qquad \|A_p^*A_{p'}\|\leqq k(p,p')^2.$$

If $h(p,p')$ *and* $k(p,p')$ *satisfy the estimates*

$$\int_{R^{2n}}h(p,p')dp\leqq M,\qquad \int_{R^{2n}}k(p,p')dp\leqq M,$$

then we have the estimate (3.3) *in Proposition* 3.3.

Sketch of the proof of Proposition 3.3. We shall prove that A_p in (3.2) satisfy the conditions of Lemma 3.4. The adjoint operator $A_{p'}^*$ of $A_{p'}$ is given by

$$A_{p'}^*v(\xi)=\int_{R^n}e^{-i\overline{S(y,\xi)}}\,\overline{a_{p'}(y,\xi)}v(y)dy.$$

Let $H_{p,p'}(x,y)$ denote the integral kernel function of $A_pA_{p'}^*$. Then

$$H_{p,p'}(x,y)=\int_{R^n}e^{i(S(x,\xi)-\overline{S(y,\xi)})}a_p(x,\xi)\overline{a_{p'}(y,\xi)}d\xi. \tag{3.4}$$

We introduce a differential operator of order 1

$$L=\rho^{-2}(1-i\min\{\lambda(p),\ \lambda(p')\}^2\,\nabla_\xi(\overline{S(x,\xi)}-S(y,\xi))\cdot\nabla_\xi),$$

where

$$\rho=(1+\min\{\lambda(p),\ \lambda(p')\}^2\,|\nabla_\xi(S(x,\xi)-\overline{S(y,\xi)})|^2)^{1/2}.$$

Since $Le^{i(S(x,\xi)-\overline{S(y,\xi)})}=e^{i(S(x,\xi)-\overline{S(y,\xi)})}$, we integrate (3.4) by parts and for $m=0,1,2,\cdots$, we have

$$H_{p,p'}(x,y)=\int_{R^n} e^{i(S(x,\xi)-\overline{S(y,\xi)})}({}^tL)^m(a_p\overline{a_{p'}})d\xi,$$

where ${}^tL$ denotes the formal transposed operator of L. By induction as Lemma 2.5 in [1] we have the estimate.

$$|({}^tL)^m(a_p\overline{a_{p'}})|\leqq C_m\rho^{-m}.$$

On the other hand we have the following estimate.

Lemma 3.5 ([1, Lemma 2.1]). *There exists a positive constant* δ_1 *such that*

$$|\nabla_\xi S_R(x,\xi)-\nabla_\xi S_R(y,\xi)|\geqq\delta_1\,|x-y|.$$

Hence we can prove the following estimate for $H_{p,p'}(x,y)$.

Lemma 3.6. *For any non-negative integer* m *we have*

$$|H_{p,p'}(x,y)|\leqq C\,|a|_m^2\chi_R\Big(\frac{\sigma-\sigma'}{\lambda(p)+\lambda(p')}\Big)\chi_R\Big(\frac{x-s}{\lambda(p)^{-1}}\Big)$$
$$\times\chi_R\Big(\frac{y-s'}{\lambda(p')^{-1}}\Big)\frac{\min\{\lambda(p),\,\lambda(p')\}^n}{(1+\min\{\lambda(p),\,\lambda(p')\}^2\,|x-y|^2)^{m/2}},$$

where χ_R *denotes the characteristic function of the ball* $\{x\,;|x|\leqq R\}$.

Using the Schur's lemma and Lemma 3.6, we have the following

Lemma 3.7. 1) *If* $|\sigma-\sigma'|\geqq R(\lambda(p)+\lambda(p'))$, *then* $A_pA_{p'}^*=0$.

2) *If* $|\sigma-\sigma'|\leqq 2R(\lambda(p)+\lambda(p'))$, *then we have the estimate*:

(i) *If* $|s-s'|\leqq 2R(\lambda(p)^{-1}+\lambda(p')^{-1})$, *then*

$$\|A_pA_{p'}^*\|\leqq C\,|a|_m^2.$$

(ii) *If* $2R(\lambda(p)^{-1}+\lambda(p')^{-1})\leqq|s-s'|\leqq 2R(\varphi(p)+\varphi(p'))$, *then*

$$\|A_pA_{p'}^*\|\leqq C\,|a|_m^2\,(1+\min\{\lambda(p),\,\lambda(p')\}^2\,|s-s'|^2)^{-m/2}.$$

(iii) *If* $2R(\varphi(p)+\varphi(p'))\leqq|s-s'|$, *then*

$$\|A_pA_{p'}^*\|\leqq C\,|a|_m^2\,(1+\min\{\Phi(p),\,\Phi(p')\}\,|s-s'|)^{-m/2}.$$

We can prove the estimate for $h(p,p')$ in Lemma 3.4 from Lemma 3.7. In doing so we note that in case of (i) or (ii) of 2) in Lemma 3.7 we have

$$\Phi(p)\approx\Phi(p'),\ \varphi(p)\approx\varphi(p')\ \text{and thus}\ \lambda(p)\approx\lambda(p').$$

In case of (iii) of 2) in Lemma 3.7 we distinguish two cases: $\Phi(p)\leqq\Phi(p')$ and $\Phi(p)\geqq\Phi(p')$ and use the estimate (iii) of (2.1) in Definition 2.1.

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