58. On the Solvability of Nonlinear Goursat Problems

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1. **Introduction.** In the study of Cauchy-Goursat problems for nonlinear analytic equations, the spectral condition, that is the condition of the minimum radius for the exceptional set of a parameter, has been a well accepted starting point (cf. [1]). Outside this exceptional disk, the problems have unique analytic solution, but no general results are known as to the geometric structure of the exceptional set inside the disk or its relation to the equation itself. In this paper, we shall give an existence-uniqueness theorem inside the disk with the description of the exceptional set, together with an alternative theorem for Goursat problems as an immediate consequence.

The theorem is proved by applying the contraction principle on some suitable Banach space. But the essential point of the arguments here lies in the proof of the existence and continuity of the inverse of the linearized operator L (cf. (2.2)), which will be shown by solving boundary value problems for difference equations.

2. **Notations and results.** Let us consider the nonlinear analytic Cauchy-Goursat problem

(2.1) $$\varepsilon D^{\beta}u=a(x, D^{\alpha}u), \qquad u=O(x^{\beta})$$

in a neighborhood of the origin of C^d, where $|\alpha|\leqq|\beta|$, $\alpha\neq\beta$ and ε is a complex constant. We assume that in the expression $a(x, D^{\alpha}u)$ the function $a(x, y_{\alpha})$ satisfies the compatibility condition $a(0,0)=0$ if $\varepsilon=0$.

We assume

(A.I) If $d\geqq 3$, there exist integers l_1 and l_2 $(1\leqq l_1<l_2\leqq d)$ such that for every α in (2.1) with $|\alpha|=|\beta|$, either that $\alpha_\nu=\beta_\nu$ for $\nu\neq l_1, l_2$ $1\leqq\nu\leqq d$ or $(\partial a/\partial y_\alpha)(0,0)=0$ holds. Here $\alpha=(\alpha_1,\cdots,\alpha_d)$, $\beta=(\beta_1,\cdots,\beta_d)$. Without loss of generality we may assume $l_1=1$, $l_2=2$. We define a linear operator L associated to (2.1) by

(2.2) $$L=\varepsilon D^{\beta}-\sum_{|\alpha|=|\beta|}(\partial a/\partial y_\alpha)(0,0)D^{\alpha}.$$

Then, setting $\lambda=\xi_1/\xi_2$ the characteristic equation for L is given by

(2.3) $$\varepsilon-\sum_{p=-m}^{n}a_p\lambda^p=0 \ (a_p=(\partial a/\partial y_\alpha)(0,0) \qquad \text{if } \alpha-\beta=(p,-p,0,\cdots,0))$$

for some integers $m, n\geqq 0$. Since the case $a_p=0$ ($\forall p>0$ or $\forall p<0$) is solved (cf. [2]) we are interested in the case where $a_n a_{-m}\neq 0$ for some $m\geqq 1$ and $n\geqq 1$.

We put

$$C_r=\left\{\varepsilon\in C\,;\ \varepsilon=\sum_{-m\leqq p\leqq n}a_p\lambda^p,\ |\lambda|=r\right\}$$

and define the set D by

$$D=\bigcup_{r>0}D_r,\qquad D_r=\left\{\varepsilon\in C\,;\ \oint_{C_r}(\eta-\varepsilon)^{-1}d\eta=0\right\}. \tag{2.4}$$

We define E_1 as the set of ε such that eq. (2.3) has at least one set of multiple roots. And let us denote by $\hat{a}(x, y_\alpha)$, the non-principal part of $a(x, y_\alpha)$:

$$\hat{a}(x, y_\alpha)=a(x, y_\alpha)-\sum_{|\alpha|=|\beta|}(\partial a/\partial y_\alpha)(0, 0)y_\alpha. \tag{2.5}$$

In case ε is in E_1 we assume the following:

(A.II) If ε is in E_1, the non-principal part $\hat{a}(x, y_\alpha)$ does not contain the variable y_α for all α satisfying that $-\hat{p}(\hat{p}-1)<|\alpha|-|\beta|\leqq 0$ where $\hat{p}$ is the largest multiplicity of the roots of eq. (2.3).

Theorem. *Suppose* (A.I) *and* (A.II). *Then there exists a countable set E_2 which does not accumulate in $D\backslash E_1$ such that for every $\varepsilon\in D\backslash E_2$ eq.* (2.1) *has one and only one holomorphic solution in a neighborhood of the origin of C^d.*

Remarks. a) We can also show that if $\varepsilon\in D\cap E_2$ eq. (2.1) may not have a solution.

b) The set E_2 is contained in the set:

$$\{\varepsilon\in C\,;\ |\varepsilon|\leqq\inf\sum|a_\alpha\xi^{\alpha-\beta}|\ \text{and}\ 2|\mathrm{Re}\,\varepsilon|\leqq\inf\sum|a_\alpha\xi^{\alpha-\beta}+\overline{a_{2\beta-\alpha}\xi^{\beta-\alpha}}|\ \text{and}\ 2|\mathrm{Im}\,\varepsilon|\leqq\inf\sum|a_\alpha\xi^{\alpha-\beta}-\overline{a_{2\beta-\alpha}\xi^{\beta-\alpha}}|\}$$

where the infimum's in the above are taken when every ξ_j $(1\leqq j\leqq d)$ ranges over $C\backslash 0$ and the sum's are taken for all α such that $\alpha\neq\beta$, $|\alpha|=|\beta|$. Here $a_\gamma=(\partial a/\partial y_\gamma)(0, 0)$.

Let the operator L be given by (2.2) and let us consider the linearized Goursat problem (L. G); $Lu=h(x)$, $u=O(x^\beta)$. Then as an immediate consequence of the above theorem we get the following:

Corollary. *Suppose* (A.I). *Then* (2.1) *has a unique analytic solution for any non-principal part $\hat{a}(x, D^\alpha u)$ satisfying* (A.II) *iff the linearized problem* (L. G) *has at least one solution for every polynomial $h(x)$ or iff the analytic solution of* (L. G) *is unique for every polynomial $h(x)$.*

References

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