57. Singular Hadamard's Variation of Domains and Eigenvalues of the Laplacian. II

By Shin OZAWA

Department of Mathematics, University of Tokyo

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§1. This paper is a continuation of our previous note [2]. Let Ω be a bounded domain in $\mathbf{R}^n$ with C^3 boundary γ and w be a fixed point in Ω. For any sufficiently small $\varepsilon>0$, let B_ε be the ball defined by $B_\varepsilon=\{z\in\Omega\,;\ |z-w|<\varepsilon\}$. Let Ω_ε be the bounded domain defined by $\Omega_\varepsilon=\Omega\setminus\bar{B}_\varepsilon$. Then the boundary of Ω_ε consists of γ and ∂B_ε. Let $0>\mu_1(\varepsilon)>\mu_2(\varepsilon)>\cdots$ be the eigenvalues of the Laplacian with the Dirichlet condition on $\partial\Omega_\varepsilon$. We arrange them repeatedly according to their multiplicities. In [2], [3] we gave the asymptotic formulas for the j-th eigenvalue $\mu_j(\varepsilon)$ when $\varepsilon\searrow 0$ in case $n=2, 3$. In this note we treat the case $n=4$. We have the following

Theorem 1. *Assume $n=4$. Fix j. Suppose that the j-th eigenvalue μ_j of the Laplacian with the Dirichlet condition on γ is a simple eigenvalue, then*

$$\mu_j(\varepsilon)-\mu_j=-2S_4\varepsilon^2\varphi_j(w)^2+O(\varepsilon^{5/2}) \tag{1.1}$$

holds when ε tends to zero. Here φ_j denotes the eigenfunction of the Laplacian with the Dirichlet condition on γ satisfying

$$\int_\Omega \varphi_j(x)^2dx=1.$$

Here S_4 denotes the area of the unit sphere in $\mathbf{R}^4$.

Our aim of this note is to offer a rough sketch of the proof of the above theorem. Calculation and technique which are used to prove Theorem 1 are more elaborate than in case $n=2$ and 3. $L^p(1<p<\infty)$ spaces are used in this note. We employed only L^2 spaces in case $n=2, 3$.

We review a generalization of the Schiffer-Spencer formula. See [6]. Also see [3]. In the following we assume $n=4$. Let $G(x,y)$ be the Green's function on Ω. Put

$$\omega_\varepsilon=\{x\in\Omega\,;\ G(x,w)<(2S_4\varepsilon^2)^{-1}\}$$

and $\beta_\varepsilon=\Omega\setminus\bar{\omega}_\varepsilon$. Let $G_\varepsilon(x,y)$ be the Green's function in ω_ε.

Variational formula for the Green's function [3]. Fix $x, y\in\Omega\setminus\{w\}$ such that $x\neq y$. Then

$$G_\varepsilon(x,y)=G(x,y)-2S_4\varepsilon^2G(x,w)G(y,w)+O(\varepsilon^3) \tag{1.2}$$

holds when ε tends to zero. The remainder term is not uniform with respect to x, y.

To prove Theorem 1 we use the iterated kernel $G_\varepsilon^{(2)}$ (resp. $G^{(2)}$) of

$G_\varepsilon(x, y)$ (resp. $G(x, y)$) and a variational formula for $G_\varepsilon^{(2)}(x, y)$. See § 2. It should be remarked that in case $n=2$, 3 only G_ε, G were used. See [3].

There are many related papers and topics. For example, see [1], [2], [4], [5] and [7]. Details and a further generalization of this note will be given elsewhere.

§ 2. **Outline of proof of Theorem 1.** Since $G(x, w)-(2\mathcal{S}_4|x-w|^2)^{-1}$ is bounded when x tends to w, we see there exists $C>0$ such that

$$\omega_{\varepsilon+C\varepsilon^3}\subset\Omega_\varepsilon\subset\omega_{\varepsilon-C\varepsilon^3}$$

holds for any small $\varepsilon>0$. Since the eigenvalues of the Laplacian with the Dirichlet condition depend monotonically on the domain, we can easily deduce Theorem 1 from the following

Proposition 1. *Assume $n=4$. Let $0>\tilde{\mu}_1(\varepsilon)>\cdots\geq\tilde{\mu}_j(\varepsilon)\geq\cdots$ be the eigenvalues of the Laplacian with the Dirichlet condition on $\partial\omega_\varepsilon$. We arrange them repeatedly according to their multiplicities. Fix j. If μ_j is a simple eigenvalue then*

$$\tilde{\mu}_j(\varepsilon)-\mu_j=-2\mathcal{S}_4\varepsilon^2\varphi_j(w)^2+O(\varepsilon^{5/2}) \tag{2.1}$$

holds when ε tends to zero.

We introduce various operators. For $\varepsilon>0$, let $G_\varepsilon^{(2)}(x, y)$ be the kernel of the operator G_ε^2 defined by

$$(G_\varepsilon^2 g)(x)=\int_{\omega_\varepsilon} G_\varepsilon^{(2)}(x, y)g(y)dy \qquad x\in\omega_\varepsilon. \tag{2.2}$$

Let G (resp. G^2) be the Green operator (resp. its iterated operator) given by

$$(Gf)(x)=\int_\Omega G(x, y)f(y)dy \tag{2.3}$$

$$\text{(resp. } (G^2 f)(x)=\int_\Omega G^{(2)}(x, y)f(y)dy).$$

To get Proposition 1 we compare the eigenvalues of G_ε^2 and G^2.

To interpolate G_ε^2 and G^2 we introduce two operators H_ε with of $\tilde{H}_\varepsilon$ given by the following:

$$(H_\varepsilon g)(x)=\int_{\omega_\varepsilon} h_\varepsilon(x, y)g(y)dy,$$

where

$$h_\varepsilon(x, y)=G^{(2)}(x, y)-2\mathcal{S}_4\varepsilon^2(G^{(2)}(x, w)G(y, w)+G(x, w)G^{(2)}(y, w))$$

for x, $y\in\omega_\varepsilon$.

$$(\tilde{H}_\varepsilon f)(x)=\int_\Omega \tilde{h}_\varepsilon(x, y)f(y)dy,$$

where

$$\tilde{h}_\varepsilon(x, y)=G^{(2)}(x, y)-2\mathcal{S}_4\varepsilon^2(G^{(2)}(x, w)G(y, w)\hat{\Psi}_\varepsilon(y) + \hat{\Psi}_\varepsilon(x)G(x, w)G^{(2)}(y, w))$$

for x, $y\in\Omega$. Here $\hat{\Psi}_\varepsilon\in C^\infty(\mathbf{R}^4)$ is defined as $\hat{\Psi}_\varepsilon(x)=1$ on $|x-w|\geq\varepsilon/2$ and $\hat{\Psi}_\varepsilon(x)=0$ on $|x-w|\leq\varepsilon/4$.

We give the following

Theorem 2.

$$\|G_\varepsilon^2 - H_\varepsilon\|_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{5/2}$$

for some constant C independent of $\varepsilon > 0$.

Our proof of Theorem 2 is rather involved, thus we put off its sketch untill § 3.

We study $\tilde{H}_\varepsilon$. We have the following

Proposition 2. *For any fixed $\varepsilon > 0$, $\tilde{H}_\varepsilon$ is a compact selfadjoint operator in $L^2(\Omega)$.*

It should be remarked that the perturbation family $\varepsilon \mapsto \tilde{H}_\varepsilon$ is not an analytic family of selfadjoint operators, thus some techniques are necessary to study eigenvalues of $\tilde{H}_\varepsilon$. Let λ_j be a simple eigenvalue of G. We construct an approximate eigenvalue of $\tilde{H}_\varepsilon$ which tends to λ_j^2 when $\varepsilon \searrow 0$. We solve the following equation for $\tilde{\varphi}_\varepsilon$.

$$(2.4) \quad (G^2 - \lambda_j^2)\tilde{\varphi}_\varepsilon(x) = -2\lambda_j^2 \varphi(w)\varphi(x)(G(\hat{\Psi}_\varepsilon \cdot \varphi))(w) + G^{(2)}(x, w)(G(\hat{\Psi}_\varepsilon \cdot \varphi))(w) + \lambda_j^2 G(x, w)\hat{\Psi}_\varepsilon(x)\varphi(w).$$

Here φ denotes the eigenfunction of the Laplacian associated with λ_j and satisfying $\|\varphi\|_{L^2(\Omega)} = 1$. The above equation is solvable since the right-side of (2.4) is orthogonal to the kernel space spanned by φ and $G^2 - \lambda_j^2$ is the Fredholm operator. We have the following

Lemma 1. *Put $r(\varepsilon) = 2S_4\varepsilon^2(G(\hat{\Psi}_\varepsilon \cdot \varphi))(w)$, then*

$$(2.5) \quad (\tilde{H}_\varepsilon - (\lambda_j^2 - 2\lambda_j^2 r(\varepsilon)))(\varphi + 2S_4\varepsilon^2\tilde{\varphi}_\varepsilon^0) = -4S_4^2\varepsilon^4(G^{(2)}(x, w)(G(\hat{\Psi}_\varepsilon \cdot \varphi))(w) + G(x, w)\hat{\Psi}_\varepsilon(x)(G^2\tilde{\varphi}_\varepsilon^0)(w) - 2\lambda_j^2\varphi(w)\tilde{\varphi}_\varepsilon^0(x)(G(\hat{\Psi}_\varepsilon \cdot \varphi))(w)),$$

where $\tilde{\varphi}_\varepsilon^0$ is the unique solution of (2.4) orthogonal to φ.

From (2.4) we easily get

$$(2.6) \quad \|\tilde{\varphi}_\varepsilon^0\|_{L^2(\Omega)} \leq C|\log \varepsilon|^{1/2}$$

for some constant C independent of $\varepsilon > 0$. By Lemma 1 and (2.6) we have the following

Lemma 2. *$L^2(\Omega)$-norm of the left-side of (2.5) does not exceed $C\varepsilon^4|\log \varepsilon|$. And $\|\varphi + 2S_4\varepsilon^2\tilde{\varphi}_\varepsilon^0\|_{L^2(\Omega)} \geq 1$.*

From Lemma 2 we can deduce the existence result for an approximate eigenvalue. We get the following

Proposition 3. *There exists at least one eigenvalue $\tilde{\lambda}_j^{(2)}(\varepsilon)$ of $\tilde{H}_\varepsilon$ satisfying*

$$\tilde{\lambda}_j^{(2)}(\varepsilon) = \lambda_j^2 - 2\lambda_j^3(2S_4)\varepsilon^2\varphi(w)^2 + O(\varepsilon^4|\log \varepsilon|).$$

We compare $\tilde{H}_\varepsilon$ with H_ε. Let ψ_ε be the eigenfunction of $\tilde{H}_\varepsilon$ with respect to $\tilde{\lambda}_j^{(2)}(\varepsilon)$. Assume $\|\psi_\varepsilon\|_{L^2(\Omega)} = 1$. We put $\psi_{\varepsilon,1} = \chi_\varepsilon\psi_\varepsilon$, $\psi_{\varepsilon,2} = (1 - \chi_\varepsilon)\psi_\varepsilon$ where χ_ε is the characteristic function of ω_ε. Then these equations are equivalent to the following equations:

$$(2.7) \quad (H_\varepsilon - \tilde{\lambda}_j^{(2)}(\varepsilon))\psi_{\varepsilon,1}(x) = \int_{\beta_\varepsilon} \tilde{h}_\varepsilon(x, y)\psi_{\varepsilon,2}(y)dy \qquad x \in \omega_\varepsilon$$

$$(2.8)\qquad \int_{\omega_\varepsilon}\tilde{h}_\varepsilon(x,y)\psi_{\varepsilon,1}(y)dy+\int_{\beta_\varepsilon}\tilde{h}_\varepsilon(x,y)\psi_{\varepsilon,2}(y)dy=\tilde{\lambda}_j^{(2)}(\varepsilon)\psi_{\varepsilon,2}\qquad x\in\beta_\varepsilon$$

$$(2.9)\qquad \|\psi_{\varepsilon,1}\|^2_{L^2(\omega_\varepsilon)}+\|\psi_{\varepsilon,2}\|^2_{L^2(\beta_\varepsilon)}=1.$$

By (2.7) and arguments from which we deduce Theorem 2 we have

$$\|(H_\varepsilon-\tilde{\lambda}_j^{(2)}(\varepsilon))\psi_{\varepsilon,1}\|_{L^2(\omega_\varepsilon)}\le C\varepsilon^{4/p}\|\psi_{\varepsilon,2}\|_{L^{p'}(\beta_\varepsilon)}$$

for any fixed p satisfying $1<p<2$. Here p' is the conjugate number of p. On the other hand we get from (2.8)

$$\|\psi_{\varepsilon,2}\|_{L^{p'}(\beta_\varepsilon)}\le C\varepsilon^{4/p'}$$

for $2<p'<\infty$, and

$$\|\psi_{\varepsilon,2}\|_{L^2(\beta_\varepsilon)}\le C\varepsilon^2.$$

Summing up these facts we have the following

Proposition 4. *There exists a constant C independent of ε such that*

$$\|(H_\varepsilon-\tilde{\lambda}_j^{(2)}(\varepsilon))\psi_{\varepsilon,1}\|_{L^2(\omega_\varepsilon)}\le C\varepsilon^4$$

and

$$\|\psi_{\varepsilon,1}\|_{L^2(\omega_\varepsilon)}\ge 1/2$$

hold.

Thus we get the following

Proposition 5. *There exists at least one eigenvalue $\lambda_*^{(2)}(\varepsilon)$ of H_ε satisfying $\lambda_*^{(2)}(\varepsilon)=\lambda_j^2-2\lambda_j^3(2S_4)\varepsilon^2\varphi(w)^2+O(\varepsilon^4|\log\varepsilon|)$.*

Now we prove Proposition 1. Let λ_j be as above. Then the following is known. See [5], [3].

Lemma 3. *Let V be a sufficiently small fixed neighbourhood of λ_j. Then there exists $\varepsilon_0>0$ such that only one simple eigenvalue $\lambda_j(\varepsilon)$ of G_ε is in V for any fixed ε satisfying $\varepsilon_0>\varepsilon>0$. Moreover $\lim_{\varepsilon\to 0}\lambda_j(\varepsilon)=\lambda_j$.*

From Theorem 3, Proposition 4 and Lemma 3 we get

$$\lambda_*^{(2)}(\varepsilon)-(\lambda_j(\varepsilon))^2=O(\varepsilon^{5/2})$$

and

$$\lambda_j(\varepsilon)^2=\lambda_j^2-2\lambda_j^3(2S_4)\varepsilon^2\varphi_j(w)^2+O(\varepsilon^{5/2}).$$

Then we have Proposition 1.

§3. Rough sketch of a proof of Theorem 2. We need two Lemmas.

Lemma 4. *If u_ε satisfies*

$$\begin{cases}\Delta u_\varepsilon=0 & in\ \omega_\varepsilon\\ u_\varepsilon|_{\partial\Omega}=0, & |u_\varepsilon|_{\partial\beta_\varepsilon}|\le H(\varepsilon)\end{cases}$$

then $|u|\le CH(\varepsilon)\varepsilon^2|x-w|^{-2}$ for $x\in\omega_\varepsilon$. C is a constant independent of ε.

Lemma 5. *If u_ε satisfies*

$$\begin{cases}\Delta^2 u_\varepsilon=0 & in\ \omega_\varepsilon\\ u_\varepsilon|_{\partial\Omega}=\Delta u_\varepsilon|_{\partial\Omega}=0 & \\ |u_\varepsilon|_{\partial\beta_\varepsilon}|\le M(\varepsilon), & |\Delta u_\varepsilon|_{\partial\beta_\varepsilon}|\le N(\varepsilon)\end{cases}$$

then

$$\|u_\varepsilon\|_{L^{8/3}(\omega_\varepsilon)} \le C(N(\varepsilon)\varepsilon^2 + M(\varepsilon)\varepsilon^{3/2})$$

holds for some constant C independent of ε.

Sketch of proof of Theorem 2. Fix $f \in C_0^\infty(\omega_\varepsilon)$. And we put $u_\varepsilon = (G_\varepsilon^2 - H_\varepsilon)f$. Then we have $\Delta^2 u_\varepsilon = 0$ in ω_ε and $u_\varepsilon|_{\partial\Omega} = \Delta u_\varepsilon|_{\partial\Omega} = 0$. To estimate $L^{8/3}(\omega_\varepsilon)$-norm of u_ε, we need bounds for $M(\varepsilon)$ and $N(\varepsilon)$ in Lemma 5.

Since we have

$$|u_\varepsilon|_{\partial\beta_\varepsilon}| \le \int_{\omega_\varepsilon} |G^{(2)}(x, y) - G^{(2)}(y, w)|_{x\in\partial\beta_\varepsilon} |f(y)| dy + CG^{(2)}(x, w)|_{x\in\partial\beta_\varepsilon} \varepsilon^2 |(Gf)(w)|$$

and

$$|\Delta u_\varepsilon|_{\partial\beta_\varepsilon}| \le \int_{\omega_\varepsilon} |G(x, y) - G(y, w)|_{x\in\partial\beta_\varepsilon} |f(y)| dy,$$

we can take $M(\varepsilon)$, $N(\varepsilon)$ as

$$M(\varepsilon) = \tilde{C} |\log \varepsilon| \varepsilon^2 \|f\|_{L^{8/3}(\omega_\varepsilon)} + \tilde{C}\varepsilon \|f\|_{L^2(\omega_\varepsilon)}$$
$$N(\varepsilon) = \tilde{C}\varepsilon^{1/2} \|f\|_{L^{8/3}(\omega_\varepsilon)}$$

for a constant $\tilde{C}$ independent of ε. Therefore we have

$$\|G_\varepsilon^2 - H_\varepsilon\|_{L^{8/3}(\omega_\varepsilon)} \le C\varepsilon^{5/2}.$$

Since we have

$$\|(G_\varepsilon^2 - H_\varepsilon)^*\|_{L^{8/5}(\omega_\varepsilon)} = \|G_\varepsilon^2 - H_\varepsilon\|_{L^{8/3}(\omega_\varepsilon)}$$

and

$$(G_\varepsilon^2 - H)^*|_{C_0^\infty(\omega_\varepsilon)} = G_\varepsilon^2 - H_\varepsilon,$$

we get Theorem 2.

Errata in [1]. The right-side of the formula in Theorem 1 in [1] should be replaced by

$$G(x, y) - (m-n-2) S_{m-n} \varepsilon^{m-n-2} \int_N G(x, w) G(y, w) dw + O(\varepsilon^{m-n}).$$

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