53. The Structure of Open Algebraic Surfaces and Its Application to Plane Curves

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The purpose of this note is to outline our recent results on the structure of algebraic surfaces which may not be complete. Details will be published elsewhere.

1. A triple $(X, \bar{X}, D)$ is said to be a non-singular triple, if $\bar{X}$ is a complete non-singular surface over the field of complex numbers and if D is a divisor with only simple normal crossings such that $X=\bar{X}\setminus D$. We denote by $K(\bar{X})$ the canonical divisor on $\bar{X}$. We define logarithmic m-genera $\bar{P}_m(X)$ and the logarithmic Kodaira dimension $\bar{\kappa}(X)$ by

$$\bar{P}_m(X)=\dim H^0(\bar{X}, m(K(\bar{X})+D)),$$
$$\bar{\kappa}(X)=\kappa(K(\bar{X})+D, \bar{X})$$

(see [2]).

In general, let Δ be a divisor on $\bar{X}$ with $\kappa(\Delta, \bar{X})\geq 0$. Then one has a $\boldsymbol{Q}$-divisor Δ^+ and an effective $\boldsymbol{Q}$-divisor Δ^- such that

(1) $\Delta=\Delta^+ +\Delta^-$,

(2) Δ^+ is semipositive (i.e. $(\Delta^+, \Gamma)\geq 0$ for all curves Γ on $\bar{X}$),

(3) the intersection matrix of Δ^- is negative-definite or $\Delta^-=0$,

(4) $(\Delta^+, \Delta^-)=0$.

This decomposition is unique and is called the Zariski decomposition of Δ (see [4] or [5]).

The main results are summarized as follows:

Theorem 1. *If $\bar{\kappa}(X)=0$, then $\bar{P}_i(X)=1$ for some i, $1\leq i\leq 66$.*

Theorem 2. *If $\bar{\kappa}(X)\geq 0$ and if D is connected, then $\bar{P}_{12}(X)>0$.*

We shall outline proofs of these theorems. A triple $(X, \bar{X}, D)$ is said to be almost minimal if the support of $(K(\bar{X})+D)^-$ contains no exceptional curve of the 1st kind.

Lemma 3. *Given a triple $(X, \bar{X}, D)$ with $\bar{\kappa}(X)\geq 0$, there exist an almost minimal triple $(Z, \bar{Z}, B)$ and a birational morphism $f: \bar{X}\to\bar{Z}$ having the following properties:*

(1) $B=f_*(D)$,

(2) $(K(\bar{X})+D)^+=f^*((K(\bar{Z})+B)^+)$.

By the above lemma, it suffices to prove the theorems for almost minimal triples $(X, \bar{X}, D)$. We need the following

Proposition 4. *If $(X, \bar{X}, D)$ is almost minimal, then $D-(K(\bar{X})+$ $+D)^-)$ is effective and $(\bar{X}, D-(K(\bar{X})+D)^-)$ is a relatively minimal model* *of* $(X, \bar{X}, D)$ *in the sense of Kawamata* [4].

Put $D_{\min}=D-(K(\bar{X})+D)^{-}$. By Kawamata [4], $n(K(\bar{X})+D_{\min})$ is generated by global sections for $n \gg 0$.

Firstly, we shall prove Theorem 1. Mimicking an argument in Fujita [1], we can construct a non-singular triple $(\mathcal{X}, \bar{\mathcal{X}}, \mathcal{D})$ and a morphism $\Pi : \bar{\mathcal{X}} \to \bar{X}$ such that

(1) $\mathcal{D}=\Pi^{-1}(D)$,

(2) $\Pi|_{\mathcal{X}}$ is an etale covering map,

(3) $\bar{P}_1(\mathcal{X})=1$.

Furthermore, we can assume that a cyclic group G acts on $\bar{\mathcal{X}}$ in such a way that $\mathcal{D}$ is G-invariant and $\mathcal{X}/G$ is isomorphic to X. Let σ be a generator of G. Then σ gives rise to an automorphism of $H^0(\bar{\mathcal{X}}, K(\bar{\mathcal{X}})+\mathcal{D})$ which will be denoted by σ^*. We denote the eigenvalue of σ^* by α. Then α is a primitive n-root of unity and we have $\bar{P}_n(X)=1$. On the other hand, by making use of the classification of the logarithmic $K3$ surfaces (Iitaka [3]), we obtain $n \leq 66$.

Secondly, we shall prove Theorem 2.

(A) The case in which $\bar{\kappa}(X)=0$. By a rather easy argument we can conclude that $\bar{P}_{12}(X)=1$.

(B) The case in which $\bar{\kappa}(X)=1$. By Kawamata [1], we have $\bar{P}_{12}(X)>0$.

(C) The case in which $\bar{\kappa}(X)=2$. We need the following

Proposition 5. *If* $(X, \bar{X}, D)$ *is almost minimal, then* $P_n(X) =1/2(nK(\bar{X})-[-(n-1)D_{\min}]+[D_{\min}],\ (n-1)K(\bar{X})-[-(n-1)D_{\min}] +[D_{\min}])+\chi(\mathcal{O}_{\bar{X}})+\varepsilon(n, D)$ *for* $n \geq 2$, *where* $\varepsilon(n, D)$ *is a non-negative integer.*

By the above proposition, we have $\bar{P}_{12}(X)>0$. Note that the proof of this part is somewhat complicated.

2. In this section, we shall outline a proof of the following

Theorem 6. *Let* (x, y, z) *be a system of homogeneous coordinates in* $\boldsymbol{P}^2$. *Let* $C_{a,\alpha_1,\cdots,\alpha_a}$ $(a \geq 3, \alpha_1, \cdots, \alpha_a \in \boldsymbol{C})$ *be the curve defined by*

$$\left(y^{a-1}z-\prod_{i=1}^{a}(x-\alpha_i y)\right)^a z + \sum_{i=1}^{a-1}\binom{a}{i}x^i y^{a^2+1-ai-i}\left(y^{a-1}z-\prod_{i=1}^{a}(x-\alpha_i y)\right)^i + y^{a^2+1}=0.$$

The curve $C_{a,\alpha_1,\cdots,\alpha_a}$ *has the following properties*:

(1) $C \backslash \{0:0:1\} \cong \boldsymbol{A}^1$,

(2) $\bar{\kappa}(\boldsymbol{P}^2 \backslash C)=1$.

Conversely, the curve having the properties (1) *and* (2) *is* $C_{a,\alpha_1,\cdots,\alpha_a}$ *up to projective equivalence for some* $(a, \alpha_1, \cdots, \alpha_a)$.

Let $\mu : \bar{X} \to \boldsymbol{P}^2$ be a composition of blow-ups such that $D=\mu^{-1}(C)$ has only simple normal crossings. We can assume that μ is shortest among such birational morphisms. We set $X=\boldsymbol{P}^2 \backslash C$. We denote by $\Pi : \bar{X} \to \Delta$

a morphism associated with $|n(K(\bar{X})+D)|$ for $n \gg 0$. Since X is affine, we have $\Pi(D)=\Delta$. This implies a general fiber of $\Pi|_X$ is $\boldsymbol{G}_m^1$. Hence, there exist a Hirzebruch surface $\bar{Y}$ and a birational morphism $\rho: \bar{X} \to \bar{Y}$ such that $\Pi \cdot \rho^{-1}$ is a morphism. We put $\varphi = \Pi \cdot \rho^{-1}$ and denote by f a general fiber of φ.

By taking a suitable $\bar{Y}$, we can assume that $\Gamma = \rho_*(D)$ is either

(i) a sum of a 2-section (i.e. an irreducible curve ε with $(\varepsilon, f)=2$) and at most three fibers, or

(ii) a sum of two sections for the fibration φ and at most three fibers.

Note that

(1) each irreducible component of D has a negative self-intersection number and

(2) the exceptional curve in D is unique.

It follows from (1) and (2) that Γ is a sum of two sections and three fibers.

Then, we can conclude that $(X, \bar{X}, D)$ is a resolution of $C_{a,\alpha_1,\cdots,\alpha_a}$.

References

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