49. *On the Mean Value Property of Harmonic and Complex Polynomials*

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1. Introduction. Throughout this note K denotes either the field of complex numbers C or the field of real numbers R. Let n be a fixed integer >2, and θ denote the number $\exp(2\pi i/n)$.

In 1935 S. Kakutani and M. Nagumo [1], and independently, in 1936 J. L. Walsh [3] proved the following theorems concerning the mean value property (MVP) of harmonic and complex polynomials.

Theorem A (Kakutani-Nagumo-Walsh). *If $f: C\to R$ is continuous, the* MVP

$$\sum_{\nu=0}^{n-1} f(x+\theta^{\nu}y)=nf(x)$$

holds for all $x, y\in C$ if, and only if, $f(x)$ is a harmonic polynomial of degree at most $n-1$.

Theorem B. *An entire function f satisfies the* MVP *for all $x, y\in C$ if and only if f is given by a complex polynomial of degree at most $n-1$.*

The above theorems are direct or indirect motivations for the generalizations and applications of various papers.

The main purpose of this note is to inform some more generalizations of Theorems A and B from the standpoint of the theory of finite difference functional equations.

2. The general solution. Definition. A mapping $Q^p: C\to K$ is called a homogeneous polynomial of degree p if and only if there exists a p-additive symmetrical mapping $Q_p: C^p\to K$; that is, $Q_p(x_1,\cdots,x_p)=Q_p(x_{i_1},\cdots,x_{i_p})$ for all $x_1,\cdots,x_p\in C$ and for all permutations $(i_1,\cdots,i_p)$ of the sequence $(1,\cdots,p)$ and Q_p is an additive function in each x_q, $1\le q\le p$, such that $Q^p(x)=Q_p(x,\cdots,x)$ for all $x\in C$. We say that Q_p is associated with Q^p or that Q_p generates Q^p.

We agree that for $p=0$ a homogeneous polynomial of degree zero is a constant.

Definition. Let β be any non-negative integer. If $f: C\to K$ is a finite sum $f=Q^0+Q^1+\cdots+Q^\beta$ of homogeneous polynomials, then f is called a generalized polynomial of degree at most β.

Notation. Let $Q_{(n-r,r)}(x\,;y)$ denote the value of $Q_n(x_1,\cdots,x_n)$ for $x_i=x$, $i=1,\cdots,n-r$ and $x_i=y$, $i=n-r+1,\cdots,n$. In particular

$Q_{(0,n)}(y\,;x)=Q_{(n,0)}(x\,;y)=Q^n(x)$.

Theorem 1. *A function $f: C\to K$ satisfies the* MVP *for all $x, y\in C$ if and only if there exists a generalized polynomial of degree at most $n-1$ such that*

$$(*)\qquad f(x)=Q^0+Q^1(x)+\cdots+Q^{n-1}(x)\qquad \textit{for all } x\in C,$$

where the homogeneous polynomials $Q^p: C\to K$ for $p=1,\cdots,n-1$ must satisfy the equation

$$\sum_{\nu=0}^{n-1}\sum_{\delta=1}^{n-1}\sum_{\sigma=1}^{\delta}\binom{\delta}{\sigma}Q_{(\delta-\sigma,\sigma)}(x\,;6^\nu y)=0\qquad \textit{for all } x, y\in C.$$

For $f: C\to K$ and for $y\in C$ we define the usual difference operator Δ_y by $\Delta_y f(x)=f(x+y)-f(x)$. For $y_i\in C$, $i=1,2,\cdots,n$, we inductively define the n-th order difference operator $\Delta^n_{y_1\cdots y_n}$ by

$$\Delta^n_{y_1\cdots y_n}f(x)=(\Delta^{n-1}_{y_1\cdots y_{n-1}})\Delta_{y_n}f(x).$$

Notice that the ring of operators generated by this family of operators is commutative and distributive.

The proof of Theorem 1 is based on Lemma 1 and Fundamental Theorem below. Let G and H be additive Abelian groups. Let S be any field and G, H be unital S-modules. Let $f: G\to H$ satisfy the equation

$$\sum_{i=0}^{n}\gamma_i f(x+\alpha_i y)=0\qquad \text{for all } x, y\in G,$$

where $n>2$ is a given integer, $\gamma_i\neq 0$, $\alpha_i\neq 0$ $(=\alpha_0)$ for $i=0,1,\cdots,n$ are fixed elements in S and $\alpha_j\neq\alpha_k$ for $j\neq k$. The above equation is a generalization of the well-known difference functional equation

$$\Delta^n_y f(x)=0,\qquad \text{i.e., } \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(x+iy)=0$$

for all $x, y\in G$. More generally we have

Lemma 1. *Let $f_i: G\to H$ for $i=0,1,\cdots,n$ satisfy the equation*

$$(**)\qquad \sum_{i=0}^{n}f_i(x+\alpha_i y)=0\qquad \textit{for all } x, y\in G,$$

where $\alpha_i\neq 0$ for $i=0,1,\cdots,n$ are fixed elements in S and $\alpha_j\neq\alpha_k$ for $j\neq k$. Then equation $(**)$ implies

$$\Delta^n_u f_i(x)=0$$

for each $i=0,1,\cdots,n$ and for all $x, u\in G$.

The following general theorem of S. Mazur and W. Orlicz [2] in the theory of finite difference functional equations plays a fundamental role in our study.

Fundamental Theorem. *Let M, N be fixed integers ≥ 0. Let X be an Abelian additive semigroup with unit element 0 and $mx=x+x+\cdots+x$ for integer $m>0$, $x\in X$, and let F be an Abelian group and $my=y+y+\cdots+y$ for integer $m>0$, $y\in F$. Let $f: X\to F$. The following three statements are equivalent if $M^N\neq 0$ in F:*

(a) $\Delta^{N+1}_y f(x)=0$ *for all $x, y\in X$,*

(b) $\Delta^{N+1}_{y_1\cdots y_{N+1}} f(x)=0$ *for all* $x, y_1, \cdots, y_{N+1} \in X$,

(c) *f is a generalized polynomial of degree at most N, that is, $f(x)=Q^0+Q^1(x)+\cdots+Q^N(x)$ for all $x \in X$, where $Q^p: X \to F$ for $p=0, 1, \cdots, N$ are homogeneous polynomials.*

3. Solutions bounded on a set of positive measure. Theorem 2. *If a function $f: C \to R$ satisfies the* MVP *for all $x, y \in C$, then $(*)$ holds for all $x \in C$, where $Q^p: C \to R$ for $p=0, 1, \cdots, n-1$. Moreover, f is bounded on a set of positive Lebesgue measure if and only if f is given by a harmonic polynomial of degree at most $n-1$.*

Theorem 3. *If a function $f: C \to C$ satisfies the* MVP *for all $x, y \in C$, then $(*)$ holds for all $x \in C$. Further, f is bounded on a set of positive Lebesgue measure if and only if f is a complex polynomial of the form*

$$f(x)=\sum_{s=0}^{n-1} a_{0,s}x^s+\sum_{r=1}^{n-1} a_{r,r}\bar{x}^r,$$

where $\bar{x}$ denotes the conjugate of x.

The detailed proofs of the above results stated in this note will be published in the Pacific J. Math.

References

[1] S. Kakutani and M. Nagumo: About the functional equation $\sum_{\nu=0}^{n-1} f(z+e^{(2\nu\pi/n)i}\xi)=nf(z)$. Zenkoku Shijô Danwakai, **66**, 10–12 (1935) (in Japanese).

[2] S. Mazur and W. Orlicz: Grundlegende Eigenschaften der polynomischen Operationen. Studia Math., **5**, 50–68 (1934).

[3] J. L. Walsh: A mean value theorem for polynomials and harmonic polynomials. Bull. Amer. Math. Soc., **42**, 923–930 (1936).