38. On the Asymptotic Behavior of Asymptotically Nonexpansive Semi-Groups in Banach Spaces

By Masataka TANIGUTI,*) Yoshitaka TAKAHASHI,**) and Hiroshi ABE***)

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1. Introduction and statement of results. Throughout this paper X denotes a *uniformly convex* real Banach space and C is a nonempty *closed* subset of X. Let J be an unbounded subset of $[0, \infty)$ such that

(1.1) $$t+s \in J \quad \text{for every } t, s \in J,$$

and

(1.2) $$t-s \in J \quad \text{for every } t, s \in J \text{ with } t>s.$$

A family $\{T(t): t \in J\}$ of mappings from C into itself is called an *asymptotically nonexpansive semi-group* on C if

(1.3) $$T(t+s)=T(t)T(s) \quad \text{for } t, s \in J$$

and there exists a function $a: J \to [0, \infty)$ with $\lim_{t\to\infty} a(t)=1$ such that

(1.4) $$\|T(t)x-T(t)y\| \leq a(t)\|x-y\| \quad \text{for every } x, y \in C \text{ and } t \in J.$$

In particular if $a(t)\equiv 1$, then $\{T(t): t \in J\}$ is called a *nonexpansive semi-group* on C. The set of fixed points of $\{T(t): t \in J\}$ will be denoted by F, i.e. $F=\{x \in C: T(t)x=x \text{ for all } t \in J\}$. We denote by $C_{11}[0, \infty)$ ($C_1[0, \infty)$) the set of increasing (nondecreasing) continuous functions defined on $[0, \infty)$.

In this paper we deal with the strong convergence of trajectories of semi-groups. Our first result is the following which extends and unifies several results in [1], [2], [4].

Theorem 1. *Let $\{T(t): t \in J\}$ be an asymptotically nonexpansive semi-group on C with $F \neq \phi$, and let $x \in C$. Suppose that*

(a_1) *there exist $x_0 \in F$, $\varphi \in C_{11}[0, \infty)$, $\psi \in C[0, \infty)$ and a nonnegative function b defined on J with $\lim_{h\to\infty} b(h)=1$ such that*

$$\varphi(\|T(h)u+T(h)v-2x_0\|) \leq \varphi(b(h)\|u+v-2x_0\|)+[\psi(b(h)\|u-x_0\|) - \psi(\|T(h)u-x_0\|)+\psi(b(h)\|v-x_0\|)-\psi(\|T(h)v-x_0\|)]$$

for every $u, v \in \{T(t)x: t \in J\}$ and $h \in J$ and

(a_2) $$\lim_{t\to\infty} \|T(t+h)x-T(t)x\|=0 \quad \textit{for every } h \in J.$$

Then $\{T(t)x: t \in J\}$ converges strongly as $t\to\infty$ to an element of F.

Remark. Suppose that $T: C\to C$ is nonexpansive (i.e. $\|Tu-Tv\|$

*) Department of Mathematics, Waseda University, Tokyo.
**) Musashino Electrical Communication Laboratory, N.T.T.
***) Yonaizawa High School, Akita.

$\leq\|u-v\|$ for $u, v\in C$), $0\in C$, $T0=0$ and there exists $c\geq 0$ such that

(1.5) $$\|Tu+Tv\|^2\leq\|u+v\|^2+c\{\|u\|^2-\|Tu\|^2+\|v\|^2-\|Tv\|^2\}\quad\text{for all } u, v\in C.$$

(This condition has been considered in [3]. Note that (1.5) with $c=0$ is satisfied if T is odd.) Then the nonexpansive semi-group $\{T^n : n=1, 2, \cdots\}$ satisfies condition (a_1) with $x_0=0$, $\varphi(t)=t^2$, $\psi(t)=ct^2$ and $b(n)\equiv 1$.

Corollary. *Let C be a closed convex subset of X, $T: C\to C$ be a nonexpansive mapping with $F(T)\neq\phi$, where $F(T)$ is the set of fixed points of T, and let $x\in C$. Suppose that*

(1.6) *there exist $x_0\in F(T)$, convex functions $\varphi\in C_{11}\,[0,\infty)$ and $\psi\in C_1\,[0,\infty)$ such that*
$$\varphi(\|Tu+Tv-2x_0\|)\leq\varphi(\|u+v-2x_0\|)+[\psi(\|u-x_0\|)-\psi(\|Tu-x_0\|)+\psi(\|v-x_0\|)-\psi(\|Tv-x_0\|)]$$
for every $u, v\in C$.

Then for each $\lambda\in(0,1)$ $\{((1-\lambda)I+\lambda T)^n x\}$ converges strongly as $n\to\infty$ to an element of $F(T)$, where I is the identity.

Let $A\subset X\times X$ be accretive. It is well known that if $R(I+\lambda A)\supset\overline{D(A)}$ for sufficiently small $\lambda>0$, then there exists a nonexpansive semi-group $\{T(t): t\geq 0\}$ on $\overline{D(A)}$ such that $T(t)x=\lim_{\lambda\to 0+}(I+\lambda A)^{-[t/\lambda]}x$ for $t\geq 0$, $x\in\overline{D(A)}$ and $T(t)x: [0,\infty)\to X$ is continuous for every $x\in\overline{D(A)}$. (See [5].) We say that $\{T(t): t\geq 0\}$ is the nonexpansive semi-group generated by A.

Let μ be a gauge function, i.e. $\mu\in C_{11}\,[0,\infty)$ with $\mu(0)=0$ and $\lim_{t\to\infty}\mu(t)=\infty$. We define F_μ (duality mapping with gauge function μ) and $\langle\ ,\ \rangle_\mu$ by

$$F_\mu(u)=\{f\in X^* : (u,f)=\|u\|\cdot\|f\| \text{ and } \|f\|=\mu(\|u\|)\}\quad\text{for } u\in X$$

and

$$\langle v, u\rangle_\mu=\sup\{(v,f): f\in F_\mu(u)\}\qquad\text{for } u, v\in X.$$

It is easily seen that $\langle v,u\rangle_\mu\leq\|v\|\cdot\mu(\|u\|)$, $\langle\alpha u+v,u\rangle_\mu=\alpha\|u\|\cdot\mu(\|u\|)+\langle v,u\rangle_\mu$ for real α, and A is accretive if and only if $\langle y-v, x-u\rangle_\mu\geq 0$ for every $[x,y]$, $[u,v]\in A$.

As applications of Theorem 1 we obtain the following theorems:

Theorem 2. *Let $A\subset X\times X$ be an accretive operator with $A^{-1}0\neq\phi$ such that $R(I+\lambda A)\supset\overline{D(A)}$ for $\lambda\in(0,\lambda_0)$, and let $x\in\overline{D(A)}$. Suppose that*

(b) *there exist $x_0\in A^{-1}0$, gauge functions φ_0, ψ_0, and a continuous function $k_0: (0,\infty)\times(0,\infty)\to[0,\infty)$ such that if*
$$[x_i, y_i]\in\{[J_\lambda^n x, (I-J_\lambda)J_\lambda^{n-1}x/\lambda] : n\geq 1 \text{ and } 0<\lambda<\lambda_0\},\quad x_i\neq x_0\ (i=1,2),$$
then
$$\langle y_1+y_2, x_1+x_2-2x_0\rangle_{\varphi_0}+k_0(\|x_1-x_0\|,\|x_2-x_0\|)\sum_{i=1}^{2}\langle y_i, x_i-x_0\rangle_{\psi_0}\geq 0$$

Let $\{T(t): t\geq 0\}$ be the nonexpansive semi-group on $\overline{D(A)}$ generated by A. If $\lim_{t\to\infty}\|T(t+h)x-T(t)x\|=0$ for every $h>0$, then $\{T(t)x:$

$t\geq 0\}$ *converges strongly as* $t\to\infty$ *to an element of* F.

Theorem 3. *Let* $A\subset X\times X$ *be an accretive operator with* $A^{-1}0\neq\phi$ *such that* $R(I+\lambda A)\supset\overline{D(A)}$ *for* $\lambda\in(0, \lambda_0)$ *and let* $x\in\overline{D(A)}$. *If* (b) *is satisfied, then for each* $\lambda\in(0, \lambda_0)$ $\{J_\lambda^n x\}$ *converges strongly as* $n\to\infty$ *to an element of* $A^{-1}0$.

Remark. Theorem 2 is an extension of Gripenberg's theorem [6, Theorem 1]. In fact, his condition ((1.4) in [6]) implies (b) with $\varphi_0(t)=\psi_0(t)=t$ since $\{[J_\lambda^n x, (I-J_\lambda)J_\lambda^{n-1}x/\lambda]: n\geq 1$ and $0<\lambda<\lambda_0\}\subset A$. Our condition (b) is satisfied if A is an odd operator with $R(I+\lambda A)\supset\overline{D(A)}$ for $\lambda\in(0, \lambda_0)$ or the subdifferential of a convex function given in [8, Theorem], [6, Proposition 2]. So Theorem 3 is an extension of [4, Theorem 2.4 (a)].

2. Proofs of theorems. **Proof of Theorem 1.** Put $d=\lim_{t\to\infty}\|T(t)x-x_0\|$. The conclusion is trivial when $d=0$. Now let $d>0$. It follows from (a_2) that

(2.1) $$\lim_{t\to\infty}\|T(t+h)x+T(t)x-2x_0\|=2d \qquad \text{for every } h\in J.$$

By (a_1) with $u=T(t)x$ and $v=T(s)x$ we have

$$\begin{aligned}\varphi(\|T(t+h)x+T(s+h)x-2x_0\|)&\leq\varphi(b(h)\|T(t)x+T(s)x-2x_0\|)\\&+[\psi(b(h)\|T(t)x-x_0\|)-\psi(\|T(t+h)x-x_0\|)\\&+\psi(b(h)\|T(s)x-x_0\|)-\psi(\|T(s+h)x-x_0\|)] \quad \text{for } t, s, h\in J.\end{aligned}$$

Letting $h\to\infty$, (2.1) and $\lim_{h\to\infty}b(h)=1$ imply

$$\begin{aligned}\varphi(2d)\leq\varphi(\|T(t)x+T(s)x-2x_0\|)&+[\psi(\|T(t+h)x-x_0\|)\\&+\psi(\|T(s)x-x_0\|)-2\psi(d)],\end{aligned}$$

and hence $\varphi(2d)\leq\liminf_{t,s\to\infty}\varphi(\|T(t)x+T(s)x-2x_0\|)$; while $\limsup_{t,s\to\infty}\varphi(\|T(t)x+T(s)x-2x_0\|)\leq\lim_{t,s\to\infty}\varphi(\|T(t)x-x_0\|+\|T(s)x-x_0\|)=\varphi(2d)$. So

$$\lim_{t,s\to\infty}\varphi(\|T(t)x+T(s)x-2x_0\|)=\varphi(2d)$$

and then

(2.2) $$\lim_{t,s\to\infty}\|T(t)x+T(s)x-2x_0\|=2d.$$

Therefore, by uniform convexity of X and $\lim_{t\to\infty}\|T(t)x-x_0\|=d>0$ we have $\lim_{t,s\to\infty}\|T(t)x-T(s)x\|=0$, whence $\{T(t)x: t\in J\}$ converges strongly as $t\to\infty$. Clearly the limit is an element of F. Q.E.D.

Proof of Corollary. Let $0<\lambda<1$ and set $T_\lambda=(1-\lambda)I+\lambda T$. Clearly $T_\lambda: C\to C$ is a contraction and $T_\lambda x_0=x_0$. Since

$$\|T_\lambda u+T_\lambda v-2x_0\|\leq\lambda\|Tu+Tv-2x_0\|+(1-\lambda)\|u+v-2x_0\|,$$

(1.6) and the convexity of φ and ψ imply

$$\begin{aligned}\varphi(\|T_\lambda u+T_\lambda v-2x_0\|)&\leq\lambda\varphi(\|Tu+Tv-2x_0\|)+(1-\lambda)\varphi(\|u+v-2x_0\|)\\&\leq\varphi(\|u+v-2x_0\|)+\lambda[\psi(\|u-x_0\|)-\psi(\|Tu-x_0\|)\\&\quad+\psi(\|v-x_0\|)-\psi(\|Tv-x_0\|)]\\&\leq\varphi(\|u+v-2x_0\|)+[\psi(\|u-x_0\|)-\psi(\|T_\lambda u-x_0\|)\\&\quad+\psi(\|v-x_0\|)-\psi(\|T_\lambda v-x_0\|)]\\&\qquad\qquad\text{for every } u, v\in C.\end{aligned}$$

Therefore the nonexpansive semi-group $\{T_\lambda^n: n=1, 2, \cdots\}$ on C satisfies condition (a_1) with $b(n)\equiv 1$. Since $\lim_{n\to\infty}\|T_\lambda^{n+1}x-T_\lambda^n x\|=0$ (see [7]), it follows from Theorem 1 that $\{T_\lambda^n x\}$ converges strongly to a point in $F(T_\lambda)=F(T)$. Q.E.D.

Proof of Theorem 2. Since $A^{-1}0\subset F$, there exists a constant $d\geq 0$ such that $\lim_{t\to\infty}\|T(t)x-x_0\|=2d$. The conclusion is trivial when $d=0$. Now let $d>0$. By virtue of Theorem 1 it suffices to show that (a_1) is satisfied. To this end we define M, φ, ψ by

$$M=\sup\{k_0(\xi,\eta):\xi,\eta\in[d,\|x-x_0\|]\}, \tag{2.3}$$

$$\varphi(t)=\int_0^t\varphi_0(s)ds \qquad \text{for } t\geq 0, \tag{2.4}$$

$$\psi(t)=M\int_0^t\psi_0(s)ds \qquad \text{for } t\geq 0. \tag{2.5}$$

Clearly $\varphi\in C_{11}[0,\infty)$ and $\psi\in C[0,\infty)$. We fix arbitrary numbers t_1, $t_2\geq 0$ and $h>0$. Since $J_\lambda x_0=x_0$ for $\lambda>0$ and $T(t)x=\lim_{\lambda\to 0+}J_\lambda^{[t/\lambda]}x$ for $t\geq 0$, we can choose $\lambda_1\in(0,\lambda_0)$ such that if $\lambda\in(0,\lambda_1)$ then

$$d\leq\|J_\lambda^{k+[t_i/\lambda]}x-x_0\|\leq\|x-x_0\| \quad \text{for } 0\leq k\leq[h/\lambda] \quad\text{and}\quad i=1,2. \tag{2.6}$$

Let $0<\lambda<\lambda_1$ and $0\leq k\leq[h/\lambda]$, and put $l_i=[t_i/\lambda]$. Noting $J_\lambda^{k+l_i}x\neq x_0$ (by (2.6)), it follows from (b) that

$$\begin{aligned}0\leq&\left\langle\sum_{i=1}^2(J_\lambda^{k+l_i-1}x-J_\lambda^{k+l_i}x),\ \sum_{i=1}^2 J_\lambda^{k+l_i}x-2x_0\right\rangle_{\varphi_0}\\&+k_0(\|J_\lambda^{k+l_1}x-x_0\|,\ \|J_\lambda^{k+l_2}x-x_0\|)\\&\times\left[\sum_{i=1}^2\langle J_\lambda^{k+l_i-1}x-J_\lambda^{k+l_i}x,\ J_\lambda^{k+l_i}x-x_0\rangle_{\psi_0}\right].\end{aligned}$$

Moreover $\langle J_\lambda^{k+l_i-1}x-J_\lambda^{k+l_i}x,\ J_\lambda^{k+l_i}x-x_0\rangle_{\psi_0}\geq 0$ by the accretivity of A and $0\in Ax_0$, and $k_0(\|J_\lambda^{k+l_1}x-x_0\|,\ \|J_\lambda^{k+l_2}x-x_0\|)\leq M$ by (2.6). Consequently,

$$\begin{aligned}0\leq&\left\langle\sum_{i=1}^2(J_\lambda^{k+l_i-1}x-J_\lambda^{k+l_i}x),\ \sum_{i=1}^2 J_\lambda^{k+l_i}x-2x_0\right\rangle_{\varphi_0}\\&+M\left[\sum_{i=1}^2\langle J_\lambda^{k+l_i-1}x-J_\lambda^{k+l_i}x,\ J_\lambda^{k+l_i}x-x_0\rangle_{\psi_0}\right]\\\leq&\left(\left\|\sum_{i=1}^2 J_\lambda^{k+l_i-1}x-2x_0\right\|-\left\|\sum_{i=1}^2 J_\lambda^{k+l_i}x-2x_0\right\|\right)\varphi_0\left(\left\|\sum_{i=1}^2 J_\lambda^{k+l_i}x-2x_0\right\|\right)\\&+M\sum_{i=1}^2(\|J_\lambda^{k+l_i-1}x-x_0\|-\|J_\lambda^{k+l_i}x-x_0\|)\psi_0(\|J_\lambda^{k+l_i}x-x_0\|)\\\leq&\int_{\|\Sigma_{i=1}^2 J_\lambda^{k+l_i}x-2x_0\|}^{\|\Sigma_{i=1}^2 J_\lambda^{k+l_i-1}x-2x_0\|}\varphi_0(\xi)\,d\xi+M\int_{\|J_\lambda^{k+l_i}x-x_0\|}^{\|J_\lambda^{k+l_i-1}x-x_0\|}\psi_0(\xi)\,d\xi\\=&\varphi\left(\left\|\sum_{i=1}^2 J_\lambda^{k+l_i-1}x-2x_0\right\|\right)-\varphi\left(\left\|\sum_{i=1}^2 J_\lambda^{k+l_i}x-2x_0\right\|\right)\\&+\sum_{i=1}^2[\psi(\|J_\lambda^{k+l_i-1}x-x_0\|)-\psi(\|J_\lambda^{k+l_i}x-x_0\|)].\end{aligned}$$

Adding these inequalities for $k=1,2,\cdots,[h/\lambda]$ and recalling $l_i=[t_i/\lambda]$ $(i=1,2)$, we have

$$\begin{aligned}\varphi\left(\left\|\sum_{i=1}^2 J_\lambda^{[h/\lambda]+[t_i/\lambda]}x-2x_0\right\|\right)\leq&\varphi\left(\left\|\sum_{i=1}^2 J_\lambda^{[t_i/\lambda]}x-2x_0\right\|\right)\\&+\sum_{i=1}^2[\psi(\|J_\lambda^{[t_i/\lambda]}x-x_0\|)-\psi(\|J_\lambda^{[h/\lambda]+[t_i/\lambda]}x-x_0\|)].\end{aligned} \tag{2.7}$$

Since $\lim_{\lambda\to 0+} J_\lambda^{[h/\lambda]+[t_i/\lambda]}x = T(h+t_i)x$, by letting $\lambda\to 0+$ in (2.7) we obtain

$$\varphi\left(\left\|\sum_{i=1}^{2} T(t_i+h)x-2x_0\right\|\right)\le\varphi\left(\left\|\sum_{i=1}^{2} T(t_i)x-2x_0\right\|\right)+\sum_{i=1}^{2}[\psi(\|T(t_i)x-x_0\|) - \psi(\|T(t_i+h)x-x_0\|)].$$

Thus $\{T(t): t\ge 0\}$ satisfies (a_1) with $b(h)\equiv 1$. Q.E.D.

Proof of Theorem 3. Let $\lambda\in(0,\lambda_0)$ and put $d=\lim_{n\to\infty}\|J_\lambda^n x-x_0\|$. The conclusion is trivial when $d=0$. Let $d>0$, then

$$0<d\le\|J_\lambda^n x-x_0\|\le\|x-x_0\| \qquad \text{for } n\ge 0.$$

We define M, φ, ψ by (2.3)–(2.5). In the similar way of obtaining (2.7), we have

$$\varphi\left(\left\|\sum_{i=1}^{2} J_\lambda^{n+l_i}x-2x_0\right\|\right)\le\varphi\left(\left\|\sum_{i=1}^{2} J_\lambda^{l_i}x-2x_0\right\|\right)+\sum_{i=1}^{2}[\psi(\|J_\lambda^{l_i}x-x_0\|) - \psi(\|J_\lambda^{l_i+n}x-x_0\|)] \qquad \text{for } l_1, l_2, n\ge 0.$$

Thus the nonexpansive semi-group $\{J_\lambda^n: n\ge 1\}$ on $\overline{D(A)}$ satisfies condition (a_1) with $b(n)\equiv 1$. Moreover $\lim_{n\to\infty}\|J_\lambda^{n+1}x-J_\lambda^n x\|=0$ by [4, Corollary 1.1 and Proposition 2.1]. So the conclusion is obtained from Theorem 1. Q.E.D.

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