28. A Note on a Conjecture of K. Harada

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Let G be a finite group and p be prime number. Let $\{\chi_1, \cdots, \chi_s\}$ be the set of all irreducible complex characters of G. For a subset J of the index set $\{1, \cdots, s\}$, we put $\{\chi_J\}=\{\chi_j\,;\,j \in J\}$ and $\rho_J=\sum_{j\in J}\chi_j(1)\chi_j$.

In [1], K. Harada stated the following;

Conjecture A. *If $\rho_J(x)=0$ for any p-singular element x of G, then $\{\chi_J\}$ is a union of p-blocks of G.*

And he proved that if a Sylow-subgroup of G is cyclic, then Conjecture A holds. In this note we prove the conjecture in the following another case.

Theorem. *If G is p-solvable, then Conjecture* A *holds.*

Proof. Assume that $\{\chi_J\}$ satisfies the condition of Conjecture A. As in [1], we may assume that $\{\chi_J\}\subseteq B$, for some p-block B of G. So we need to show $\{\chi_J\}=B$ or $\{\chi_J\}=\phi$.

By rearranging the index set if necessary, we may assume that $B=\{\chi_1, \cdots, \chi_k\}$. Let $\{\varphi_1, \cdots, \varphi_l\}$ be the set of all irreducible Brauer characters in B and $\{\Phi_1, \cdots, \Phi_l\}$ be the set of all principal indecomposable characters in B. For $x \in G$, we define $\boldsymbol{\chi}_B(x)$ to be the column vector of size k whose i-th component is $\chi_i(x)$. For $1\leq m\leq l$, let $\boldsymbol{d}_m$ be the column of size k whose i-th component d_{im} is the decomposition number of χ_i with respect to φ_m. Then we have

$$\boldsymbol{\chi}_B(x)=\sum_{m=1}^{l}\boldsymbol{d}_m\varphi_m(x) \qquad \text{for any } p\text{-regular element } x.$$

In particular

$$\boldsymbol{\chi}_B(1)=\sum_{m=1}^{l}\boldsymbol{d}_m\varphi_m(1).$$

Let $\boldsymbol{\chi}_J$ be the column of size k whose i-th component a_i is defined as follows. If $i \in J$, then $a_i=\chi_i(1)$ and $a_i=0$ otherwise. At first we show that $\boldsymbol{\chi}_J$ is a linear combination of $\boldsymbol{d}_m$, $m=1, \cdots, l$. Since ρ_J vanishes on all p-singular elements of G, ρ_J is an integral linear combination of Φ_m, $m=1, \cdots, l$;

$$\rho_J=\sum_m \alpha_m\Phi_m=\sum_m \alpha_m \sum_i d_{im}\chi_i=\sum_i \Big(\sum_m \alpha_m d_{im}\Big)\chi_i.$$

By the linear independence of $\{\chi_i\}$, we obtain

$$\boldsymbol{\chi}_J=\sum_m \alpha_m \boldsymbol{d}_m.$$

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Since G is p-solvable, by Theorem of Fong and Swan ([2, p. 147]) we may assume $\chi_i=\varphi_i$ on p-regular element of G $(i=1, \cdots, l)$. So the decomposition matrix of B has the form

$$(\boldsymbol{d}_1, \cdots, \boldsymbol{d}_l)=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & * \ * & \end{pmatrix}.$$

Then we have $\alpha_m=0$ or $\varphi_m(1)$.

Let $J'=\{1, \cdots, k\}-J$. Since $\{\chi'_J\}$ satisfies the condition of Conjecture A, by the same argument we get

$$\boldsymbol{\chi}'_J=\sum_m \beta_m \boldsymbol{d}_m, \quad \beta_m=0 \quad \text{or} \quad \varphi_m(1).$$

Clearly we have $\{m\,;\beta_m\neq 0\}\cap\{m\,;\alpha_m\neq 0\}=\phi$. By the definition of blocks, we get $\{m\,;\beta_m\neq 0\}=\phi$ or $\{m\,;\alpha_m\neq 0\}=\phi$. Hence $\boldsymbol{\chi}_J=\sum_m \varphi_m(1)\boldsymbol{d}_m$ or $\boldsymbol{O}$, this completes the proof of the theorem.

References

[1] K. Harada: A conjecture and a theorem on blocks of modular representation (preprint).

[2] J. P. Serre: Représentations linéaires des groupes finis. Hermann, Paris (1971).