24. *On the Boundedness and the Attractivity Properties of Nonlinear Second Order Differential Equations*

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1. Introduction. In this paper we consider the boundedness and the attractivity properties of the forced second order nonlinear nonautonomous differential equation

(1) $(a(t)x')'+h(t,x,x')+q(t)f(x)g(x')=e(t,x,x')$.

In [2], J. R. Graef and P. W. Spikes discussed the same problems as above, under some conditions. The condition described in [2] on the perturbed term $e(t,x,x')$ implies $e(t,x,x')\equiv 0$ if $q(t)$ is independent of t. On the other hand, in [1], T. A. Burton considered the same problems as above for the equation

(2) $x''+f(x)h(x')x'+g(x)=e(t)$

under some conditions.

For the equation (1) our results are strict extensions of those obtained in [2].

The attractivity result of Theorem 2 that obtained in [1] is a special case of our result.

2. Theorems. First, we consider the boundedness of solutions of the equation

(1) $(a(t)x')'+h(t,x,x')+q(t)f(x)g(x')=e(t,x,x')$

or an equivalent system of equations

(3) $x'=y,\ y'=\dfrac{1}{a(t)}\{-a'(t)y-h(t,x,y)-q(t)f(x)g(y)+e(t,x,y)\}$.

Assumption A_1. (I) *$a(t)$ and $q(t)$ are continuously differentiable, positive functions in $I=[0,+\infty)$,*

(II) *$f(x)$ is a continuous function in R^1 which satisfies*

$$\int_0^{\pm\infty} f(x)dx=+\infty,$$

(III) *$g(y)$ is a continuous, positive function in R^1,*

(IV) *$h(t,x,y)$ and $e(t,x,y)$ are continuous functions in $I\times R^2$ and $h(t,x,y)$ satisfies the inequality $yh(t,x,y)\geqq 0$ in $I\times R^2$.*

We shall define $a'(t)_+=\max\{a'(t),0\}$ and $a'(t)_-=\max\{-a'(t),0\}$ so that $a'(t)=a'(t)_+-a'(t)_-$. We also define the functions $F(x)$ and $G(y)$ by $F(x)=\int_0^x f(u)du$ and $G(y)=\int_0^y (v/g(v))dv$.

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Theorem 1. *Suppose that Assumption* A_1 *and the following conditions hold.*

(4) $\int_0^\infty \frac{|a'(t)|}{a(t)} dt < \infty, \qquad \int_0^\infty \frac{q'(t)_-}{q(t)} dt < \infty.$

(5) $y^2/g(y) \leqq MG(y)$ *in* $|y| \geqq k$ *for some constants* $M>0$ *and* $k \geqq 0$.

(6) *There exists a continuous, nonnegative function* $r(t)$ *satisfying*

$$|e(t, x, y)| \leqq \frac{a(t)|q'(t)|}{Mq(t)} + r(t) \text{ and } \int_0^\infty r(t)dt < \infty.$$

Then all solutions of (1) *are bounded.*

If, in addition, the functions $G(y)$ *and* $q(t)$ *satisfy the condition*

(7) $G(y) \to \infty$ *as* $|y| \to \infty$ *and* $q(t) \leqq q_2$ *for some constant* q_2, *then all solutions of* (3) *are bounded.*

Remark 1. From (4), there exist positive constants a_1, a_2 and q_1 such that $a_1 \leqq a(t) \leqq a_2$ and $q_1 \leqq q(t)$ in I, because

$$a(t) = a(0)\exp\left\{\int_0^t \frac{a'(s)}{a(s)} ds\right\} \geqq a(0)\exp\left\{-\int_0^\infty \frac{a'(s)_-}{a(s)} ds\right\} = a_1,$$

$$a(t) \leqq a(0)\exp\left\{\int_0^\infty \frac{a'(s)_+}{a(s)} ds\right\} = a_2$$

$$\text{and } q(t) \geqq q(0)\exp\left\{-\int_0^\infty \frac{q'(s)_-}{q(s)} ds\right\} = q_1.$$

Condition (III) in Assumption A_1 implies that condition (5) is equivalent to the following condition:

(5)′ There exists a constant $M'>0$ such that $y^2/g(y) \leqq M'G(y)$ in R^1. Moreover it follows from condition (5) that $|y|/g(y) \leqq m + MG(y)$ and $y^2/g(y) \leqq m' + MG(y)$ in R^1 for some positive constants m and m'.

Proof of Theorem 1. Since condition (II) implies that $F(x) \to \infty$ as $|x| \to \infty$, there exists a real number F_0 satisfying the inequality $F(x) + F_0 \geqq 0$ for arbitrary x in R^1. Let

$$V(t, x, y) = \left[\frac{q(t)}{a(t)} \cdot (F(x) + F_0) + G(y) + \frac{m}{M}\right] \cdot \exp\left\{-\int_0^t \frac{a'(s)_-}{a(s)} ds + 2\int_0^t \frac{q'(s)_-}{q(s)} ds\right\}$$

and differentiate $V(t) \equiv V(t, x(t), y(t))$ with respect to t for any solution $(x(t), y(t))$ of (3), then we have for any $t \geqq 0$,

$$\begin{aligned} V'(t) \leqq & \left[\left(-\frac{a'(t)_+}{a(t)} + \frac{|q'(t)|}{q(t)}\right) \cdot \frac{q(t)}{a(t)} \cdot (F(x) + F_0) + \left(-\frac{a'(t)_-}{a(t)} + 2\frac{q'(t)_-}{q(t)}\right)\right. \\ & \left. \cdot \left(G(y) + \frac{m}{M}\right) - \frac{a'(t)y^2}{a(t)g(y)} - \frac{yh(t, x, y)}{a(t)g(y)} + \frac{ye(t, x, y)}{a(t)g(y)}\right] \\ & \cdot \exp\left\{-\int_0^t \frac{a'(s)_-}{a(s)} ds + 2\int_0^t \frac{q'(s)_-}{q(s)} ds\right\} \\ \leqq & \left\{\frac{|q'(t)|}{q(t)} + M\frac{a'(t)_-}{a(t)} + 2\frac{q'(t)_-}{q(t)} + \frac{M}{a_1} r(t)\right\} V(t) \end{aligned}$$

$$+\left\{m'\frac{a'(t)_-}{a(t)}+\frac{2mq'(t)_-}{Mq(t)}+\frac{m}{a_1}r(t)\right\}\exp\left\{2\int_0^t\frac{q'(s)_-}{q(s)}ds\right\}$$

This gives the following inequality:

$$V(t)\leqq V(t_0)+\int_{t_0}^t\left\{m'\frac{a'(s)_-}{a(s)}+\frac{2mq'(s)_-}{Mq(s)}+\frac{m}{a_1}r(s)\right\}$$
$$\cdot\exp\left\{2\int_0^s\frac{q'(\tau)_-}{q(\tau)}d\tau\right\}ds+\int_{t_0}^t\left\{\frac{|q'(s)|}{q(s)}+M\frac{a'(s)_-}{a(s)}\right.$$
$$\left.+2\frac{q'(s)_-}{q(s)}+\frac{M}{a_1}r(s)\right\}V(s)ds \qquad \text{for } t\geqq t_0\geqq 0.$$

From (4), (6) and Gronwall's lemma, we obtain

$$V(t)\leqq\left[V(t_0)+\int_{t_0}^\infty\left\{m'\frac{a'(s)_-}{a(s)}+\frac{2mq'(s)_-}{Mq(s)}+\frac{m}{a_1}r(s)\right\}ds\right.$$
$$\left.\cdot\exp\left\{2\int_0^\infty\frac{q'(s)_-}{q(s)}ds\right\}\right]\cdot\exp\left[\int_{t_0}^t\left\{\frac{|q'(s)|}{q(s)}+M\frac{a'(s)_-}{a(s)}\right.\right.$$
$$\left.\left.+\frac{2q'(s)_-}{q(s)}+\frac{M}{a_1}r(s)\right\}ds\right]$$
$$=c_1\cdot\exp\left[\int_{t_0}^t\left\{\frac{q'(s)}{q(s)}+M\frac{a'(s)_-}{a(s)}+4\frac{q'(s)_-}{q(s)}+\frac{M}{a_1}r(s)\right\}ds\right]$$
$$\leqq c_1\cdot\exp\left[\int_0^\infty\left\{M\frac{a'(s)_-}{a(s)}+\frac{q'(s)_-}{q(s)}+\frac{M}{a_1}r(s)\right\}ds\right]\cdot\frac{q(t)}{q(t_0)}$$
$$\leqq c_2q(t) \qquad \text{for } t\geqq t_0.$$

Therefore it follows that

$$F(x(t))\leqq V(t)\frac{a(t)}{q(t)}\exp\left[\int_0^t\left\{\frac{a'(s)_-}{a(s)}-2\frac{q'(s)_-}{q(s)}\right\}ds\right]$$
$$\leqq c_2a_2\exp\left\{\int_0^\infty\frac{a'(s)_-}{a(s)}ds\right\}$$

and

$$G(y(t))\leqq c_2\cdot\exp\left\{\int_0^\infty\frac{a'(s)_-}{a(s)}ds\right\}\cdot q(t) \qquad \text{for } t\geqq t_0.$$

The conclusions of Theorem 1 follow from (II) and (7). Q.E.D.

Corollary 1. *Suppose that Assumption* A_1, *condition* (6) *and the following conditions hold.*

(8) $a'(t)\geqq 0$, $\int_0^\infty\frac{q'(t)_-}{q(t)}dt<\infty$ *and* $a(t)\leqq a_2$ *for some* $a_2>0$.

(9) $|y|/g(y)\leqq m+MG(y)$ *in* R^1 *for some positive constants* m *and* M.

Then all solutions of (1) *are bounded.*

If, in addition, the functions $G(y)$ *and* $q(t)$ *satisfy condition* (7), *then all solutions of* (3) *are bounded.*

Next, we consider the attractivity properties of the equation

(10) $(a(t)x')'+p(t)f_1(x)g_1(x')x'+q(t)f_2(x)g_2(x')x=e(t,x,x')$

or an equivalent system of equations

(11) $x'=y,$

$$y'=\frac{1}{a(t)}\{-a'(t)y-p(t)f_1(x)g_1(y)y-q(t)f_2(x)g_2(y)x+e(t,x,y)\}.$$

Assumption A_2. (I) *$a(t)$ and $q(t)$ are continuously differentiable, positive functions in $I=[0,+\infty)$,*

(V) *$p(t)$ is continuous in I and satisfies $p_1\leqq p(t)\leqq p_2$ for some positive constants p_1 and p_2,*

(VI) *$f_1(x)$ and $f_2(x)$ are continuous, positive functions in R^1 and $f_2(x)$ satisfies $\int_0^{\pm\infty} xf_2(x)dx=+\infty$,*

(VII) *$g_1(y)$ and $g_2(y)$ are continuous, positive functions in R^1 and $g_2(y)$ satisfies $\int_0^{\pm\infty}\frac{y}{g_2(y)}dy=+\infty$,*

(VIII) *$e(t,x,y)$ is a continuous function in $I\times R^2$.*

We define the function $G_0(y)$ by $G_0(y)=\int_0^y \frac{v}{g_2(v)}dv$.

Theorem 2. *Suppose that Assumption A_2 and the following conditions hold.*

(12) $\int_0^\infty \frac{|a'(t)|}{a(t)}dt<\infty$ *and* $\int_0^\infty \frac{|q'(t)|}{q(t)}dt<\infty.$

(13) *$y^2/g_2(y)\leqq MG_0(y)$ in $|y|\geqq k$ for some constants $M>0$ and $k\geqq 0$.*

(14) *There exists a continuous nonnegative function $r(t)$ such that*

$$|e(t,x,y)|\leqq r(t) \text{ in } I\times R^2 \text{ and } \int_0^\infty r(t)dt<\infty.$$

Then every solution of (11) approaches $(0,0)$ as $t\to\infty$.

We require the following lemma to prove Theorem 2.

Lemma 1. *Consider the system of differential equations*

(S) $x'=f(t,x),\quad f\in C[I\times D]$ *where* $D=\{x\in R^n \mid \|x\|\leqq K\}$.

If there exists a Liapunov function $U(t,x)$ such that

(i) $U\in C^1[I\times D]$,

(ii) *$a\cdot\|x\|^2\leqq U(t,x)$ where a is a positive constant,*

(iii) *$U'_{(S)}\leqq -\lambda U+r(t)$ where λ is a positive constant, $r\in C[I]$,*

$$r(t)\geqq 0,\quad \int_0^\infty r(t)dt<\infty \quad\text{and}\quad U'_{(S)}=\frac{\partial U}{\partial t}+f\cdot\operatorname{grad} U,$$

then every solution, defined in the future in D, approaches the origin as $t\to\infty$.

Proof of Lemma 1. Let $x(t)$ be a solution of (S) which stays in D for $t\geqq t_0$ and let $U(t)=U(t,x(t))$. Then from (iii) we have that

$$U(t)\leqq U(t_0)e^{-\lambda(t-t_0)}+\int_{t_0}^t e^{-\lambda(t-s)}r(s)ds \qquad \text{for } t\geqq t_0.$$

This inequality and condition (ii) imply that

$$\|x(t)\|^2\leqq\frac{1}{a}\left\{U(t_0)e^{-\lambda(t-t_0)}+\int_{t_0}^t e^{-\lambda(t-s)}r(s)ds\right\}\to 0 \quad\text{as}\quad t\to\infty.$$

Therefore $x(t)$ approaches the origin as $t\to\infty$. Q.E.D.

Proof of Theorem 2. From condition (12) there exist positive constants a_1, a_2, q_1 and q_2 satisfying $a_1 \leqq a(t) \leqq a_2$ and $q_1 \leqq q(t) \leqq q_2$ in I. Therefore the boundedness of solutions of (11) is an immediate consequence of Theorem 1. Then for each solution $(x(t), y(t))$ defined in $[t_0, \infty)$ of (11), there exists a positive constant K such that $|x(t)|+|y(t)| \leqq K$ for $t \geqq t_0$. Now we define $F_1(x)=\int_0^x f_1(u)du$, $F_2(x)=\int_0^x uf_2(u)du$, $G_1(y)=\int_0^y \frac{1}{g_1(v)}dv$ and $G_2(y)=LG_0(y)-\frac{1}{2}\{G_1(y)\}^2$ where L is a positive constant to be determined later. Conditions (VI) and (VII) imply that

(15) $c_1 \leqq f_1(x) \leqq c_2$, $c_3 \leqq f_2(x) \leqq c_4$, $c_5 \leqq g_1(y) \leqq c_6$ and $c_7 \leqq g_2(y) \leqq c_8$ in $|x|+|y| \leqq K$ for some positive constants $c_1, c_2, \cdots, c_8$. Let

$$V(t, x, y)=\frac{1}{2q(t)}\{F_1(x)+G_1(y)\}^2+\frac{L}{a(t)}F_2(x)+\frac{1}{q(t)}G_2(y)$$

for $t \in I$, $|x|+|y| \leqq K$, then we have

$$V(t, x, y) \geqq \frac{L}{a(t)}F_2(x)+\frac{1}{q(t)}G_2(y) \geqq \frac{c_3 L}{2a_2}x^2+\frac{1}{q_2}\left(\frac{L}{2c_8}-\frac{1}{2c_5^2}\right)y^2 \geqq 0$$

for L large enough. Differentiating $V(t) \equiv V(t, x(t), y(t))$ with respect to t for any solution $(x(t), y(t))$ of (11), we obtain

$$\begin{aligned} V'(t)= & -\frac{q'(t)}{2q(t)^2}\{F_1(x)G_1(y)\}^2+\frac{1}{q(t)}f_1(x)y\{F_1(x)+G_1(y)\}-\frac{a'(t)F_1(x)y}{a(t)q(t)g_1(y)} \\ & -\frac{p(t)}{a(t)q(t)}F_1(x)f_1(x)y-\frac{F_1(x)f_2(x)g_2(y)x}{a(t)g_1(y)}-\frac{La'(t)}{a(t)^2}F_2(x) \\ & -\frac{q'(t)}{q(t)^2}G_2(y)-\frac{La'(t)y^2}{a(t)q(t)g_2(y)}-\frac{Lp(t)f_1(x)g_1(y)y^2}{a(t)q(t)g_2(y)}+\frac{e(t, x, y)}{a(t)q(t)} \\ & \times\left\{\frac{F_1(x)}{g_1(y)}+\frac{Ly}{g_2(y)}\right\} \leqq \frac{q'(t)_-}{q(t)}\left[\frac{1}{2q(t)}\{F_1(x)+G_1(y)\}^2+\frac{1}{q(t)}|G_2(y)|\right] \\ & +\frac{a'(t)_-}{a(t)}\left\{\frac{|F_1(x)y|}{q(t)g_1(y)}+\frac{L}{a(t)}F_2(x)+\frac{Ly^2}{q(t)g_2(y)}\right\} \\ & +\frac{r(t)}{a(t)q(t)}\left\{\frac{|F_1(x)|}{g_1(y)}+\frac{L|y|}{g_2(y)}\right\}+\frac{f_1(x)}{q(t)}\{|F_1(x)y|+yG_1(y)\} \\ & +\frac{p(t)f_1(x)}{a(t)q(t)}|F_1(x)y|-\frac{f_2(x)g_2(y)}{a(t)g_1(y)}xF_1(x)-\frac{Lp(t)f_1(x)g_1(y)}{a(t)q(t)g_2(y)}y^2. \end{aligned}$$

From (15), we obtain $|F_1(x)y| \leqq c_2|xy|$, $yG_1(y) \leqq (1/c_5)y^2$ and $xF_1(x) \geqq c_1 x^2$ in $|x|+|y| \leqq K$. We can also choose L so large that

$$G_2(y) \geqq \left(\frac{L}{c_8}-\frac{1}{c_5^2}\right)y^2 \geqq 0, \qquad \frac{y^2}{g_2(y)} \leqq G_2(y),$$

$$\frac{|F_1(x)|}{g_1(y)}+\frac{L|y|}{g_2(y)} \leqq \frac{c_2}{c_5}|x|+\frac{L}{c_7}|y| \leqq \left(\frac{c_2}{c_5}+\frac{L}{c_7}\right)K,$$

$$\begin{aligned} & \left\{1+\frac{p(t)}{a(t)}\right\}\frac{f_1(x)}{q(t)}\cdot|F_1(x)y|+\frac{f_1(x)}{q(t)}yG_1(y)-\frac{f_2(x)g_2(y)}{a(t)g_1(y)}xF_1(x) \\ & -\frac{Lp(t)f_1(x)g_1(y)}{a(t)q(t)g_2(y)}y^2 \leqq -c_9(x^2+y^2) \end{aligned}$$

and $c_{10}(x^2+y^2)\leqq V(t, x, y)\leqq c_{11}(x^2+y^2)$ in $|x|+|y|\leqq K$ for some positive constants c_9, c_{10}, c_{11}. It is easy to show that $|F_1(x)y|/g_1(y)\leqq(c_2/2c_5)K^2$ in $|x|+|y|\leqq K$. Thus we have the estimates

$$V'(t)\leqq\frac{q'(t)_-}{q(t)}V(t)+(1+L)\frac{a'(t)_-}{a(t)}V(t)+\frac{c_2K^2}{2c_5q_1}\cdot\frac{a'(t)_-}{a(t)}$$
$$+\frac{K}{a_1q_1}\left(\frac{c_2}{c_5}+\frac{L}{c_7}\right)r(t)-c_9(x^2+y^2)$$
$$\leqq L_1\left[\left\{\frac{q'(t)_-}{q(t)}+\frac{a'(t)_-}{a(t)}\right\}V(t)+\frac{a'(t)_-}{a(t)}+r(t)\right]-c_9(x^2+y^2)$$

for some constant $L_1>0$. Define

$$W(t, x, y)=V(t, x, y)\cdot\exp\left[-L_1\int_0^t\left\{\frac{q'(s)_-}{q(s)}+\frac{a'(s)_-}{a(s)}\right\}ds\right],$$

then we obtain

$$W(t, x, y)\geqq c_{10}\cdot\exp\left[-L_1\int_0^\infty\left\{\frac{q'(s)_-}{q(s)}+\frac{a'(s)_-}{a(s)}\right\}ds\right]\cdot(x^2+y^2)$$

and

$$W'(t)\leqq\left\{L_1\left(\frac{a'(t)_-}{a(t)}+r(t)\right)-c_9(x^2+y^2)\right\}\exp\left[-L_1\int_0^t\left\{\frac{q'(s)_-}{q(s)}+\frac{a'(s)_-}{a(s)}\right\}ds\right],$$

where $W(t, x(t), y(t))$ for any solution $(x(t), y(t))$ of (11). We will use Lemma 1 to complete the proof of Theorem 2. Q.E.D.

Remark 2. If we replace condition (6) by the following condition:

(6)′ $$|e(t,x,y)|\leqq\frac{a(t)|q'(t)|}{Mq(t)}+r_1(t)+r_2(t)|y|,\quad\int_0^\infty r_i(t)dt<\infty\ (i=1, 2),$$

then the same conclusions as those of Theorem 1 and those of Corollary 1 are valid.

Remark 3. If we replace condition (14) by

(14)′ $$|e(t, x, y)|\leqq r_1(t)+r_2(t)|y|,\quad\int_0^\infty r_i(t)dt<\infty\ (i=1, 2),$$

then the same conclusion as that of Theorem 2 is valid.

Remark 4. If we replace condition (14) by

(14)″ $$|e(t, x, y)|\leqq r_1(t)+r_2(t)|x|+r_3(t)|y|,\quad\int_0^\infty r_i(t)dt<\infty\ (i=1, 2, 3),$$

and we assume that $f_2(x)\geqq\varepsilon>0$ in R^1 and either that $y^2/g_2(y)\leqq MG_0(y)$, $g_2(y)\geqq\delta>0$ in R^1 or that $|y|/g_2(y)\leqq M\sqrt{G_0(y)}$, $g_2(y)\leqq\gamma$ in R^1, then the same conclusion as that of Theorem 2 is valid.

The proofs of these results are analogous to that of J. W. Heidel [3] and will be published later.

References

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