1. An Approximate Positive Part of Essentially Self-Adjoint Pseudo-Differential Operators. I

By Daisuke FUJIWARA
Department of Mathematics, University of Tokyo

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1981)

§0. **Introduction.** Let $a(x, \xi)$ be a real valued symbol function belonging to the class $S^1_{1,0}(\mathrm{R}^n)$ of Hörmander [6], that is, for any pair of multi-indices α and β, we have

$$\sup_{x,\xi}(1+|\xi|^2)^{(|\beta|-1)/2}|D_x^\alpha D_\xi^\beta a(x, \xi)|<\infty,$$

where we used usual multi-index notation. Let $a^w(x, D)$ denote its Weyl quantization, which is defined as

$$a^w(x, D)u(x)=\left(\frac{1}{2\pi}\right)^n\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}a\left(\frac{x+y}{2}, \xi\right)e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi$$

for any $u\in\mathcal{S}(\mathrm{R}^n)$. Cf. Weyl [11], Voros [10] and Hörmander [8].

Since $a(x, \xi)$ is real valued, the operator $a^w(x, D)$ is essentially self-adjoint in the Hilbert space $L^2(\mathrm{R}^n)$. We shall denote scalar product and norm in $L^2(\mathrm{R}^n)$ by $(\ ,\ )$ and $\|\ \|$, respectively. The main result in this note is the following

Theorem. *Let $a(x, \xi)$ be as above. Let ε be an arbitrary small positive number. Using the symbol function $a(x, \xi)$, we can construct three bounded linear operators π^+, π^- and R in $L^2(\mathrm{R}^n)$ with the following properties;*

1) *π^+ and π^- are non-negative symmetric operators.*

2) *There exists a positive constant C such that we have*

$$Re(\pi^+a^w(x, D)u, u)\geq -C\|u\|^2,$$

and

$$-Re(\pi^-a^w(x, D)u, u)\geq -C\|u\|^2$$

for any $u\in\mathcal{S}(\mathrm{R}^n)$.

3) $$\pi^+ + \pi^- = I+R,$$

where R satisfies the estimate $\|R\|<\varepsilon$ and $\|a^w(x,D)R\|+\|Ra^w(x,D)\|<\infty$.

All these operators π^+, π^- and R can be written as integral operators.

If $a(x, \xi)\geq 0$ for any $(x, \xi)\in\mathrm{R}^{2n}$, our construction shows that $\pi^+=I$ and $\pi^-=R=0$. Thus, in this case our theorem is nothing but the celebrated sharp Gårding inequality. In this case, sharper results are known in [9], [8] and in [4]. However, if $a(x, \xi)$ changes sign, very little was done (cf. [5]) and our result seems new.

It is not clear to the author whether the above result has relationship to a deeper problem:

"To what extent can one know spectral properties of $a^w(x, D)$ from local behaviours of its symbol function $a(x, \xi)$?"

Acknowledgements. The problem treated in this note was proposed by L. Nirenberg. The author wishes to express his sincere gratitude to Prof. L. Nirenberg for that.

§ 1. Micro-localization. We use a modification of the ingenious partition of unity used by Beals-Fefferman [2]. We partition $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ into rectangles $Q_j^1 = Q_{xj}^1 \times Q_{\xi j}^1$, $j=1, 2, \cdots$, obtained by partitioning the x-space into cubes of diameter 1 and partitioning the ξ-space into cubes of the diameter satisfying

$$16^{-1}(N+|\xi|) \leq \text{diam. } Q_{\xi j}^1 \leq 8^{-1}(N+|\xi|) \tag{1.1}$$

for all $(x, \xi) \in Q_j^1$. Here N is a large positive number to be fixed later. For any $r>0$, rQ_j^1 denotes the rectangle r-homothetic to Q_j^1 with the same center as Q_j^1.

We retain the rectangle Q_j^1 which satisfies any one of the following conditions;

$$a(x, \xi) \text{ has constant sign if } (x, \xi) \in 4Q_j^1. \tag{1.2}$$

$$\left|\frac{\partial}{\partial \xi_k} a(x, \xi)\right| \geq \text{diam. of } Q_{xj}^1 \text{ for } (x, \xi) \in 4Q_j^1. \tag{1.3$_k$}$$

$$\left|(1+|\xi|)^{-1}\frac{\partial}{\partial x_k} a(x, \xi)\right| \geq 2 \text{ diam. of } Q_{xj}^1 \text{ for any } (x, \xi) \in 4Q_j^1. \tag{1.4$_k$}$$

$$\text{diam. } Q_{xj}^1 < 2\, N^{1/2}(N+|\xi|)^{-1/2} \text{ for some } (x, \xi) \in Q_j^1. \tag{1.5}$$

If all these conditions fail for Q_j^1, we partition it into 2^{2n} congruent subrectangles. We denote the new rectangles $\{Q_j^2\}_j$. We retain those new rectangles which satisfy any of conditions (1.2)–(1.5) with Q_j^1 replaced by Q_j^2. We subdivide the rest. We continue this process. On any compact subset of $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$, this process ends after finite number of steps because of (1.5). When all these steps of iterative construction are complete, we relabel retained rectangles as $Q_1, Q_2, \cdots, Q_\nu = Q_{x\nu} \times Q_{\xi\nu}, \cdots$. These retained rectangles are a partition of $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ into closed sets with disjoint interiors. Let δ_ν denote the diameter of $Q_{x\nu}$ and ε_ν denote the diameter of $Q_{\xi\nu}$. In the following, we shall denote by C various positive constants independent of N, ν and h.

Proposition 1.1. *If $2Q_\mu \cap 2Q_\nu \neq \phi$, we have*

$$8^{-1}\delta_\mu \leq \delta_\nu \leq 8\delta_\mu \quad \textit{and} \quad 2^{-5}\varepsilon_\mu < \varepsilon_\nu < 2^5\varepsilon_\mu.$$

Lemma 1.2. *Let Q_μ be a rectangle. Then one of the following cases holds.*

(I) *There exists a positive constant C such that*

$$|a(x, \xi)| \leq CN^2 \quad \textit{for any} \quad (x, \xi) \in 4Q_\mu. \tag{1.6}$$

(II) *For any* $(x, \xi) \in 4Q_\mu$, $|\xi| \geq N/2$ *and* $a(x, \xi) \geq 0$.

(III) *For any* $(x, \xi) \in 4Q_\mu$, $|\xi| \geq N/2$ *and* $a(x, \xi) \leq 0$.

$(\mathrm{IV})_k$ *For any* $(x, \xi) \in 4Q_\mu$, $|\xi| \geq N/2$ *and* $\left|\frac{\partial}{\partial \xi_k} a(x, \xi)\right| \geq \delta_\mu$.

$(\mathrm{V})_k$ *For any* $(x, \xi) \in 4Q_\mu$,

$$|\xi| \geq N/2 \quad \text{and} \quad \left|\frac{\partial}{\partial x_k} a(x, \xi)\right| > \delta_\mu(N + |\xi|).$$

Let $\{\varphi_\mu(x, \xi)\}_\mu$ be non-negative C^∞ functions such that $\sum_\mu \varphi_\mu(x, \xi)^2 \equiv 1$ and $\operatorname{supp} \varphi_\mu \subset 5/4Q_\mu$. Let ϕ_μ be a non-negative C^∞ function such that $\phi_\mu(x, \xi) = 1$ on $11/8Q_\mu$ and $\phi_\mu(x, \xi) = 0$ outside $3/2Q_\mu$. We put $a_\mu(x, \xi) = a(x, \xi)\phi_\mu(x, \xi)$. We have the following estimates.

Proposition 1.3. *For any multi-indices* α *and* β, *we have*

$$|D_x^\alpha D_\xi^\beta \varphi_\mu(x, \xi)| \leq C_{\alpha\beta} \delta_\mu^{-|\alpha|} \varepsilon_\mu^{-|\beta|}. \tag{1.7}$$

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} N^{\beta^*} \delta_\mu^{1-|\alpha|} \varepsilon_\mu^{1-|\beta|} \ \textit{if}\ (x, \xi) \in 4Q_\mu. \tag{1.8}$$

where $\beta^* = \max(1, |\beta| - 1)$.

If $|\xi^\mu| \geq 2N$ *at the center* (x^μ, ξ^μ) *of* Q_μ, *we have*

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \delta_\mu^{1-|\alpha|} \varepsilon_\mu^{1-|\beta|} \ \textit{for}\ (x, \xi) \in 4Q_\mu. \tag{1.9}$$

§ 2. Solutions to the micro-localized problem. In each of cases (I)–$(\mathrm{V})_k$ of Lemma 1.2, we can prove

Lemma 2.1. *Let* $\delta_\mu^{-1} \varepsilon_\mu^{-1} = h_\mu$. *Then, for any* μ, *we can construct two symmetric bounded linear operators* π^+ *and* π^- *such that*

(i) *There exists a positive constant C such that*

$$\|\pi_\mu^+\| + \|\pi_\mu^-\| \leq C. \tag{2.1}$$

(ii) *There exists a positive constant C such that we have*

$$Re(\pi_\mu^+ a_\mu^w(x, D) \varphi_\mu^w(x, D)u, \varphi_\mu^w(x, D)u) \geq -CN^2 \|\varphi_\mu^w(x, D)u\|^2, \tag{2.2}$$

$$-Re(\pi_\mu^- a_\mu^w(x, D) \varphi_\mu^w(x, D)u, \varphi_\mu^w(x, D)u) \geq -CN^2 \|\varphi_\mu^w(x, D)u\|^2. \tag{2.3}$$

(iii) *Either* $\pi_\mu^+ + \pi_\mu^- = I$ *or* $\pi_\mu^+ + \pi_\mu^- = \phi_\mu^w(x, D) + h_\mu^2 R_\mu$, *where* R_μ *is an operator with* $\|R_\mu\| < C$ *and* $\phi_\mu \in C_0^\infty(11/8Q_\mu)$ *with* $\phi_\mu(x, \xi) = 1$ *on* $5/4Q_\mu$.

Sketch of the proof of Lemma 2.1. In the case (I) of Lemma 1.2, we put $\pi_\mu^+ = I$ and $\pi_\mu^- = 0$. Then (2.2) and (2.3) hold.

In the case (II) of Lemma 1.2, we put $\pi_\mu^+ = I$ and $\pi_\mu^- = 0$. In the case (III) of Lemma 1.2, we put $\pi_\mu^+ = 0$ and $\pi_\mu^- = I$. In the case $(\mathrm{IV})_k$ of Lemma 1.2, proof of Lemma 2.1 is rather complicated. We use the unitary operator S_μ defined by $S_\mu u(x) = \delta_\mu^{-n/2} u(\delta_\mu^{-1}(x - x^\mu)) \exp i\xi^\mu \cdot (x - x^\mu)$, where (x^μ, ξ^μ) is the center of Q_μ. Then we have

$$S_\mu^{-1} a^w(x, D) S_\mu = a^{\#w}(x, h_\mu D),$$

here $a_\mu^\#(x, \xi) = a(x^\mu + \delta_\mu x, \xi^\mu + \varepsilon_\mu \xi)$ and for any $h > 0$

$$a_\mu^\#(x, hD)u(x) = \left(\frac{1}{2\pi h}\right)^n \iint a_\mu^\#(x, \xi) e^{ih^{-1}(x-y)\cdot\xi} u(y)\, dy\, d\xi.$$

We define $\varphi_\mu^\#(x, \xi)$ and $\phi_\mu^\#(x, \xi)$ in the similar manner and we put $a\phi_\mu^\#(x, \xi) = a_\mu^\#(x, \xi)\phi_\mu^\#(x, \xi)$. We put $Q_0 = \{(x, \xi) \mid |x_j| \leq 1/n^{1/2}, |\xi_j| \leq 1/n^{1/2}, j = 1, 2, \cdots, n\}$. Then, $(x^\mu + \delta_\mu x, \xi^\mu + \varepsilon_\mu \xi) \in rQ$ if $(x, \xi) \in rQ_0$.

Lemma 2.2. $\operatorname{Supp} \varphi_\mu^\# \subset 5/4Q_0$ *and* $\operatorname{supp} \phi_\mu^\# \in 3/2Q_0$. *For any multi-indices* α *and* β, *we have the estimates*;

$$|D_x^\alpha D_\xi^\beta a^\sharp(x,\xi)|\le h_\mu^{-1} C_{\alpha\beta} \qquad \text{if } (x,\xi)\in 4Q_0$$

and

$$|D_x^\alpha D_\xi^\beta \varphi_\mu^\sharp(x,\xi)|+|D_x^\alpha D_\xi^\beta \phi_\mu^\sharp(x,\xi)\le C_{\alpha\beta} \quad \text{for any } (x,\xi)\in \mathbf{R}^n\times\mathbf{R}^n.$$

We put $b_\mu(x,\xi)=h_\mu a_\mu^\sharp(x,\xi)$. In the case of $(\mathrm{IV})_k$ of Lemma 1.2, we have $|(\partial/\partial\xi_k)b(x,\xi)|\ge 1$ for any $(x,\xi)\in 4Q_0$. This means that the Hamiltonian vector field of $b(x,\xi)$ has non-zero projection to the x-space $\mathbf{R}_x^n$. Using this, we can find local (not necessarily homogeneous) canonical transformation χ such that $b\cdot\chi(y,\eta)=\eta_k$. We can find an oscillatory integral operator $T(h)$;

$$T(h)\,u(x)=\left(\frac{1}{2\pi h}\right)^n\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} g(x,\eta)\rho(x,\eta)e^{ih^{-1}(S(x,\eta)-y\cdot\eta)}u(y)\,dy\,d\eta,$$

where $S(x,\eta)$ is a generating function of χ,

$$g(x,\eta)=\left|\det\frac{\partial^2}{\partial x\partial\eta}S(x,\eta)\right|^{-1/2}$$

and $\rho(x,\eta)$ is a cutting function (cf. [1]).

Lemma 2.3. *For any $h>0$,*

$$T(h)^*(b_\mu\phi_\mu^\sharp)^w(x,hD)-hD_kT(h)^*=hR_1(h).$$
$$T(h)T(h)^*=(\rho^2)^w(x,hD)+h^2R_2(h).$$

There exists a positive constant C such that

$$\|R_1(h)\|+\|R_2(h)\|\le C.$$

The operator $hD_k=h(1/i)(\partial/\partial x_k)$ is easily decomposed into positive part and negative part if we use the projection operators $Y^\pm(hD_k)$;

$$Y^\pm(hD_k)u(x)=\left(\frac{1}{2\pi h}\right)^n\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} Y^\pm(\eta_k)e^{ih^{-1}(x-y)\cdot\eta}u(y)\,dy\,d\eta,$$

where $Y^+(t)=1$ for $t\ge 0$ and $Y^+(t)=0$ for $t<0$ and $Y^-(t)=1-Y^+(t)$. Although the set $\{(x,\partial S(x,\eta)/\partial\eta)\,|\,\rho(x,\eta)=1\}$ is very small, a bounded number of such sets cover $5/4Q_0$ which contains $\operatorname{supp}\varphi_\mu^\sharp$. Thus we can prove

Lemma 2.4. *Assume that $(\mathrm{IV})_k$ of Lemma 1.2 holds. Then, we can construct operators $\hat\pi_\mu^+$, $\hat\pi_\mu^-$ and $\hat R_\mu(h_\mu)$ and a function $\hat\phi_\mu(x,\xi)$ such that*

(i) *$\hat\pi_\mu^\pm$ are non-negative symmetric operators. There exists a positive constant C such that $\|\hat\pi_\mu^+\|+\|\hat\pi_\mu^-\|+\|R(h_\mu)\|\le C$.*

(ii) $$Re(\hat\pi_\mu^+(b\phi_\mu^\sharp)^w(x,h_\mu D)u,u)\ge -C\,h_\mu\|u\|^2,$$
$$-Re(\hat\pi_\mu^-(b\phi_\mu^\sharp)^w(x,h_\mu D)u,u)\ge -C\,h_\mu\|u\|^2.$$

(iv) $$\hat\pi_\mu^+ +\hat\pi_\mu^- =\tilde\phi(x,h_\mu D)+h_\mu^2\tilde R_\mu(h),$$
$$\tilde\phi_\mu(x,\xi)=1 \text{ on } 5/4Q_0 \text{ and } \operatorname{supp}\tilde\phi_\mu\subset 11/8Q_0.$$

If we put $\pi_\mu^\pm=S_\mu\hat\pi_\mu^\pm S_\mu^{-1}$, we can prove that Lemma 2.4 implies Lemma 2.1 in the case of $(\mathrm{IV})_k$ of Lemma 1.2.

To prove Lemma 2.1 in the case $(\mathrm{V})_k$ of Lemma 1.2, we use Fourier transform with a parameter $h>0$;

$$F_hu(y)=\left(\frac{1}{2\pi h}\right)^n\int_{\mathbf{R}^n} e^{-ih^{-1}yx}u(x)\,dx.$$

We have

$$F_h^{-1}\, b^w(x, hD)\, F_h = p^w(y, hD),$$

where $p(y, \eta) = b_\mu(\eta, -y)$. Condition $(\mathrm{V})_k$ implies that $|\partial p(y, \eta)/\partial \eta_k| \geq 1$ for any $(y, \eta) \in 4Q_0$. Thus we can apply Lemma 2.4 with b replaced by p. Since F_h is unitary and the Legendre transform $\chi_L : (y, \eta) \to (\eta, -y)$ preserves rQ_0 for any $r > 0$, Lemma 2.1 can be proved in the case $(\mathrm{V})_k$ of Lemma 1.2.

§ 3. Patching of microlocal solutions. Collecting microlocal solution $\pi_\mu^\pm$ in Lemma 2.1, we prove our main theorem. We put

$$\pi^\pm = \sum_\mu \varphi_\mu^w(x, D) \pi_\mu^\pm\, \varphi_\mu^w(x, D). \tag{3.1}$$

Then, we have

$$\pi^+ + \pi^- = I + J_1 + J_2, \tag{3.2}$$

where

$$J_1 = \sum_\mu \{\varphi_\mu^w(x, D)\, \phi_\mu^w(x, D)\, \varphi_\mu^w(x, D) - (\varphi_\mu^2)^w(x, D)\} \tag{3.3}$$

and

$$J_2 = \sum_\mu h_\mu^2\, \varphi_\mu^w(x, D)\, R_\mu\, \varphi_\mu^w(x, D). \tag{3.4}$$

We can prove that

$$\|J_1\| + \|J_2\| \leq C\, N^{-1}. \tag{3.5}$$

Thus we take N so large that $C\,N^{-1} < \varepsilon$ and we fix N. This proves assertion 3) of Theorem. We have

$$\pi^+\, a^w(x, D) = \sum_\mu \varphi_\mu^w(x, D)\, \pi_\mu^+\, a^w(x, D)\, \varphi_\mu^w(x, D) + R_1 + R_2, \tag{3.6}$$

where

$$R_1 = \sum_\mu \varphi_\mu^w(x, D)\, \pi_\mu^+ [\varphi_\mu^w(x, D), a_\mu^w(x, D)], \tag{3.7}$$

$$R_2 = \varphi_\mu^w(x, D)\, \pi_\mu^+\, \varphi_\mu^w(x, D)\, (a^w(x, D) - a_\mu^w(x, D)). \tag{3.8}$$

Since Lemma 2.1 holds, we have only to prove that

$$\|R_1\| + \|R_2\| \leq C. \tag{3.9}$$

We prove estimates (3.5) and (3.9) by using Hörmander's theory [8]. To do so we introduce a Riemannian metric on $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$.

Definition 3.1. Let $w = (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$. We define a quadratic form g_w of $(t, \tau) \in \mathbf{R}^n \times \mathbf{R}^n$,

$$g_w(t, \tau) = \sum\nolimits_\mu \varphi_\mu(x, \xi)^2 \{\delta_\mu^{-2} |t|^2 + \varepsilon_\mu^{-2} |\tau|^2\}.$$

This is a σ-temperate Riemannian metric on $\mathbf{R}^n \times \mathbf{R}^n$ in the sense of Hörmander [8].

Lemma 3.2. *There exists a constant $C > 0$ such that for any points $w = (x, \xi)$ and $w' = (y, \eta)$;*

$$g_w^\sigma(t, \tau) \leq C\, g_{w'}^\sigma(t, \tau) \quad (1 + g_{w'}^\sigma(x - y, \xi - \eta))^3.$$

References

[1] Asada, K., and Fujiwara, D.: On some oscillatory integral transformation in $L^2(\mathrm{R}^n)$. Jap. J. Math., **4**, 299–361 (1978).

[2] Beals, R., and Fefferman, C.: On local solvability of linear partial differential equations. Ann. of Math., **27**, 1–27 (1974).

[3] Calderón, A. P., and Vaillancourt, R.: A class of bounded pseudo-differential operators. Proc. Nat. Acad. Sci. U.S.A., **69**, 1185–1187 (1971).

[4] Fefferman, C., and Phong, D. H.: On positivity of pseudo-differential operators. ibid., **75**, 4673–4674 (1978).

[5] Fujiwara, D.: An approximate positive part of a self-adjoint pseudo-differential operator. I. Osaka J. Math., **11**, 265–281 (1974); ditto. II. ibid., **11**, 283–293 (1974).

[6] Hörmander, L.: Pseudo-differential operators and hypo-elliptic equations. Proc. Symp. Pure Math., **10**, 118–196 (1966).

[7] ——: Pseudo-differential operators and non-elliptic boundary problems. Ann. of Math., **83**, 129–209 (1966).

[8] ——: The Weyl calculus of pseudo-differential operators. Comm. Pure Appl. Math., **32**, 359–443 (1979).

[9] Melin, A.: Lower bounds for pseudo-differential operators. Ark. Mat., **9**, 117–140 (1971).

[10] Voros, A.: An algebra of pseudo-differential operators and the asymptotics of quantum mechanics. J. Funct. Anal., **29**, 104–132 (1978).

[11] Weyl, H.: The Theory of Groups and Quantum Mechanics. Dover (1950).