3. *Note on Nonlinear Volterra Integral Equation in Hilbert Space*

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In [2] M. G. Crandall and J. A. Nohel showed that the initial value problem

(1) $$u'(t)+Au(t)\ni G(u)(t),\quad 0<t\leqq T,\quad u(0)=x,$$

and the Volterra equation

(2) $$u(t)+\int_0^t b(t-s)Au(s)ds\ni F(t),\qquad 0<t\leqq T$$

are equivalent so long as strong solutions of respective equations are concerned, where A is an m-accretive operator in some real Banach space X, $b\in AC([0,T];R)$, $b'\in BV([0,T];R)$, $b(0)=1$, $F\in W^{1,1}(0,T;X)$, $F(0)=x$, and

(3) $$G(u)(t)=f(t)+(r*f)(t)-r(0)u(t)+r(t)x-(u*r')(t),$$

$f=F'$, $a=b'$, $a+r+a*r=0$. Here

$$(r*f)(t)=\int_0^t r(t-s)f(s)ds,\ (u*r')(t)=\int_0^t u(t-s)dr(s).$$

In [2] the existence and uniqueness of the integral solution of (1) is shown for a more general operator G than that defined by (3). When the strong solution is considered, it is required to assume that $x\in D(A)$ or something like that so that the integral of (2) exists as a Bochner integral.

In this note we consider the case where A is the subdifferential of a proper convex lower semicontinuous function ϕ defined in a real Hilbert space X and f is such that $\int_0^T |f(t)|^2 tdt<\infty$ in addition to $f\in L^1(0,T;X)$. It will be shown that the equivalence of (1) and (2) remains valid for $x\in\overline{D(A)}$ if we interpret $b*Au$ as an improper integral.

In view of Theorem 3.6 of H. Brézis [1] the following estimate holds for the solution of (1):

$$\left(\int_0^T |u'(t)|^2 tdt\right)^{1/2}\leqq\left(\int_0^T |G(u)(t)-h|^2 tdt\right)^{1/2}+\frac{1}{\sqrt{2}}\int_0^T |G(u)(t)-h|\,dt+\frac{1}{\sqrt{2}}|u(0)-v|$$

where v and h are arbitrary elements satisfying $h\in\partial\phi(v)$. Hence in what follows w always denotes the function such that $w(t)\in Au(t)$ a.e.

and $\int_0^T |w(t)|^2 t dt < \infty$.

Under the assumptions stated above we have the following

Theorem 1. *Suppose that u is the solution of* (1) *and w is the function such that $u'(t)+w(t)=G(u)(t)$ a.e. Then for $\varepsilon>0$ $\int_\varepsilon^t b(t-s)w(s)ds$ is uniformly bounded, and converges to $F(t)-u(t)$ as $\varepsilon \to 0$ uniformly in every closed subset of the interval $(0, T]$. Conversely if the last statement is true and $u(0)=x$, then u is the solution of* (1). *In this case*

$$\int_0^t a(t-s)w(s)ds = \lim_{\varepsilon \to 0} \int_\varepsilon^t a(t-s)w(s)ds$$

exists and the following relation holds a.e. in $(0, \infty)$:

$$\frac{du(t)}{dt} + w(t) + \int_0^t a(t-s)w(s)ds = f(t).$$

As for the stability of the solution the following theorem analogous to Theorem 2 of S.-O. Londen [3] holds.

Theorem 2. *Suppose in addition to the assumptions of Theorem* 1

(4) $$b(t)>0 \quad \text{on } t \geqq 0, \quad b \in L^1(0, \infty; R),$$

(5) $$a(t) \leqq 0 \qquad \text{a.e. on } t \geqq 0,$$

(6) $$\sum_{n=0}^{\infty} \left\{ \int_n^{n+1} |f(\tau)|^2 d\tau \right\}^{1/2} < \infty.$$

Then

$$\sup_{t>1} \int_t^{t+1} |w(\tau)|^2 d\tau < \infty, \ \sup_{t>0} |u(t)| < \infty, \ u' \in L^2(0, \infty; X).$$

Proof of Theorem 1. The first part of the theorem is established rather straightforwardly by substituting

$$w(s) = f(s) + (r*f)(s) - r(0)u(s) + r(s)x - (u*r')(s) - u'(s)$$

in $\int_\varepsilon^t b(t-s)w(s)ds$ and integrating by part in an appropriate manner. Next suppose that $\int_\varepsilon^t b(t-s)w(s)ds$ is uniformly bounded and converges to $F(t)-u(t)$ uniformly in $[\delta, T]$ for every $\delta>0$, and $u(0)=x$. As is easily seen $\int_\varepsilon^t b(t-s)w(s)ds$ is absolutely continuous in $[\varepsilon, T]$. Applying Fubini's theorem and integrating by part we obtain

$$\int_\varepsilon^t a(t-s)w(s)ds = -r(0) \int_\varepsilon^t b(t-s)w(s)ds + \int_\varepsilon^t \int_\varepsilon^\tau b(\tau-s)w(s)ds dr(t-\tau).$$

This equality implies that $\int_\varepsilon^t a(t-s)w(s)ds$ is uniformly bounded and

$$\frac{d}{dt}\int_\varepsilon^t b(t-s)w(s)ds = w(t)+\int_\varepsilon^t a(t-s)w(s)ds$$

$$\to w(t)-r(0)(F(t)-u(t))+\int_0^t (F(\tau)-u(\tau))dr(t-\tau)$$

$$= w(t)+r(0)u(t)-r(t)x-(r*f)(t)+(u*r')(t)$$

at almost every $t\in(0, T]$ as $\varepsilon\to 0$. Hence for $0<t<t'$

$$\begin{aligned}F(t')-u(t')-F(t)+u(t)\\ &=\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^{t'} b(t'-s)w(s)ds-\int_\varepsilon^t b(t-s)w(s)ds\right\}\\ &=\lim_{\varepsilon\to 0}\int_t^{t'}\frac{d}{d\tau}\int_\varepsilon^\tau b(\tau-s)w(s)dsd\tau\\ &=\int_t^{t'}\{w(\tau)+r(0)u(\tau)-r(\tau)x-(r*f)(\tau)+(u*r')(\tau)\}d\tau.\end{aligned}$$

Thus u is absolutely continuous in every closed interval of $(0, T]$ and satisfies (1).

Proof of Theorem 2. Rewrite (1) as

$$u(t)+\int_1^t b(t-s)w(s)ds=F(t)-\int_0^1 b(t-s)w(s)ds$$

and consider the equation in $[1,\infty)$. If

$$\sum_{n=1}^\infty\left\{\int_n^{n+1}\left|\int_0^1 a(\tau-s)w(s)ds\right|^2 d\tau\right\}^{1/2}<\infty, \tag{7}$$

we can apply Theorem 2 of [3] to deduce the conclusion of the theorem. The relation (7) is an easy consequence of

$$\begin{aligned}\left|\int_0^1 a(t-s)w(s)ds\right| &\leqq |a(t-1)|\max_{0\leqq s\leqq 1}|G(u)(s)|\\ &+2(|a(t-1)|+|a(t)|)\max_{0\leqq s\leqq 1}|u(s)|\end{aligned}$$

and (4), (5).

References

[1] H. Brézis: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert. North-Holland and Elsevier, Amsterdam (1973).

[2] M. G. Crandall and J. A. Nohel: An abstract functional differential equation and a related nonlinear Volterra equation. Israel J. Math., **29**, 313–328 (1978).

[3] S.-O. Londen: On an integral equation in a Hilbert space. SIAM J. Math. Anal., **8**, 950–970 (1977).