109. *Riemann-Lebesgue Lemma for Real Reductive Groups*

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1. Introduction. Let G be a Lie group of class $\mathcal{H}$, which is a reductive group defined in §2. Let $P=MAN$ be a cuspidal parabolic subgroup of G and its Langlands decomposition. For any representation σ of discrete series of M and (not necessarily unitary) character λ of A, we can associate a continuous representation $\pi^{(P)}_{\sigma,\lambda}$ of G. The Fourier-Laplace transform of $f \in C_c(G)$ is defined by

$$\hat{f}_P(\sigma, \lambda)=\int_G f(x)\pi^{(P)}_{\sigma,\lambda}(x)\,dx.$$

Let V_σ be the representation space of σ. Let K be a maximal compact subgroup of G. Then $\hat{f}_P(\sigma, \lambda)$ is an integral operator on a subspace $\mathfrak{H}_\sigma$ of $L^2(K; V_\sigma)$ with the kernel function $\hat{f}_P(\sigma, \lambda; k_1, k_2)$, $k_1, k_2 \in K$. If λ is unitary, $\hat{f}_P(\sigma, \lambda)$ is defined for $L^1(G)$ and it vanishes when $(\sigma, \lambda)\to\infty$ in the sense of hull-kernel topology (see [2, p. 317]). The purpose of the present paper is to show that there exists a tube domain $\mathcal{F}^1$, containing the unitary dual A^* of A, of the complexification of A^* such that for almost all $(k_1, k_2) \in K\times K$ $\hat{f}_P(\sigma, \lambda; k_1, k_2)$ is defined for $f \in L^1(G)$ and it vanishes when $\lambda=\xi+i\eta \in \mathcal{F}^1$ and $(\sigma, \lambda)\to\infty$.

2. Notation and preliminaries. If V is a real vector space, V_c denotes its complexification. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let G^0 be the connected component of the unit of G. We denote by G_1 the analytic subgroup of G whose Lie algebra is $\mathfrak{g}_1=[\mathfrak{g}, \mathfrak{g}]$. Let G_c be the connected complex adjoint group of $\mathfrak{g}_c$. A Lie group G with Lie algebra $\mathfrak{g}$ is called of class $\mathcal{H}$ if G satisfies the following conditions: (1) $\mathfrak{g}$ is reductive and $\mathrm{Ad}(G)\subset G_c$; (2) the center of G_1 is finite; (3) $[G:G^0]<\infty$. In the sequel, we assume that G is a Lie group of class $\mathcal{H}$. If L is a Lie group, we denote by $\mathfrak{l}=\mathrm{LA}(L)$ the Lie algebra of L.

Let K be a maximal compact subgroup of G. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{s}$, $\mathfrak{k}=\mathrm{LA}(K)$, be the Cartan decomposition of $\mathfrak{g}$ and θ the corresponding Cartan involution. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{s}$ and $\mathfrak{a}_0^*$ its dual space. We denote by Δ the set of all roots of $(\mathfrak{g}, \mathfrak{a}_0)$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the corresponding root space. We fix an order in $\mathfrak{a}_0^*$ and denote by Δ^+ the set of all positive roots. We set $\mathfrak{n}_0=\sum_{\alpha\in\Delta}\oplus\mathfrak{g}_\alpha$. Let M_0 be the centralizer of $\mathfrak{a}_0$ in K. We put $A_0=\exp\mathfrak{a}_0$, $N_0=\exp\mathfrak{n}_0$

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and $P_0=M_0A_0N_0$. Then P_0 is a minimal parabolic subgroup of G. Let $\log: A_0\to\mathfrak{a}_0$ be the inverse mapping of exponential mapping of $\mathfrak{a}_0$ to A_0.

Let $P=MAN$ be a parabolic subgroup of G and its Langlands decomposition, where A is the split component and N is the radical of P. Let us assume that $P\supset P_0$. Then $A\subset A_0$. Let $\mathfrak{a}=\mathrm{LA}(A)$ and $\mathfrak{n}=\mathrm{LA}(N)$. We put $\rho_P(H)=(1/2)\mathrm{tr}(\mathrm{ad}\,H)_{\mathfrak{n}}$ for $H\in\mathfrak{a}$ and we put $\rho_0=\rho_{P_0}$. Let dk be the Haar measure on K normalized so that the total measure is one. Let dx be the standard Haar measure on G, which is the measure normalized so that $dx=e^{2\rho_P(\log a_0)}dk\,da_0\,dn_0$ for $x=ka_0n_0$ ($K\in K$, $a_0\in A_0$, $n_0\in N_0$). Let dm be the standard Haar measure on M. We put $P_M=P_0\cap M$, $K_M=K\cap M$, $A_M=A_0\cap M$ and $N_M=N_0\cap M$. Then P_M is a minimal parabolic subgroup of M and $M=K_MA_MN_M$ is an Iwasawa decomposition of M. We put $\mathfrak{a}_M=\mathrm{LA}(A_M)$ and $\mathfrak{n}_M=\mathrm{LA}(N_M)$. Let $\rho_M(^*H)=(1/2)\,\mathrm{tr}\,(\mathrm{ad}\,^*H)_{\mathfrak{n}_M}$, $^*H\in\mathfrak{a}_M$. Then if $m={}^*k{}^*a{}^*n\in K_MA_MN_M$, then $dm=e^{2\rho_M(\log{}^*a)}d^*k\,d^*a\,d^*n$. Then we have the following (see e.g. [6, p. 293]).

Lemma 1. *Each element of M commutes with every element of A. As the direct products, we have $A_0=A_MA$ and $N_0=N_MN$. If $a_0={}^*aa$ ($^*a\in A_M$, $a\in A$) and $n_0={}^*nn$ ($^*n\in N_M$, $n\in N$), then $da_0=d^*a\,da$ and $dn_0=d^*n\,dn$. Moreover, if $H_0={}^*H+H$ ($^*H\in\mathfrak{a}_M$, $H\in\mathfrak{a}$), then $\rho_0(H_0)=\rho_M(^*H)+\rho_P(H)$.*

Let us assume that P is cuspidal. Then the discrete series $\hat{M}_d$ of M is not empty. Furthermore, we assume that $M\neq G$. Then $\mathfrak{a}\neq\{0\}$. Let (σ, V_σ) be an irreducible unitary representation of M, whose class is in $\hat{M}_d$. For $\lambda\in\mathfrak{a}_c^*$ we define a representation σ_λ of P on V_σ by $\sigma_\lambda(man)=\sigma(m)e^{-i\lambda(\log a)}$, ($m\in M$, $a\in A$, $n\in N$). We put $\delta_P(man)=e^{2\rho_P(\log a)}$. Let $\pi_{\sigma,\lambda}^{(P)}$ be the representation of G induced from the representation $\delta_P^{1/2}\sigma_\lambda$ of P on V_σ. Let $\mathfrak{H}_\sigma$ be the Hilbert space consisting of all V_σ-valued measurable functions ϕ on K such that: (1) $\phi(km)=\sigma(m)^{-1}\phi(k)$ for all $m\in K_M$ and $k\in K$; (2) $\|\phi\|^2=\int_K|\phi(k)|^2dk<\infty$. Let $x=\kappa(x)m(x)\times\exp(H_P(x))n(x)$, where $\kappa(x)\in K$, $m(x)\in M$, $H_P(x)\in\mathfrak{a}$ and $n(x)\in N$. Then

$$(\pi_{\sigma,\lambda}^{(P)}(x)\phi)(k)=\sigma(m(x^{-1}k))^{-1}\,e^{(i\lambda-\rho_P)(H_P(x^{-1}k))}\phi(\kappa(x^{-1}k)),$$

($k\in K$, $\phi\in\mathfrak{H}_\sigma$). Though the components $\kappa(x)$ and $m(x)$ of x are not uniquely determined, this representation is well-defined. We know that $\pi_{\sigma,\lambda}^{(P)}$ is unitary for $\lambda\in\mathfrak{a}^*$ and that if $\lambda\in\mathfrak{a}^*$ and is regular, then $\pi_{\sigma,\lambda}^{(P)}$ is irreducible (see [4]).

3. Riemann-Lebesgue lemma. We define the Fourier transform of $f\in L^1(G)$ by

$$\hat{f}_P(\sigma, \lambda)=\int_G f(x)\pi_{\sigma,\lambda}^{(P)}(x)\,dx, \quad (\sigma\in\hat{M}_d,\ \lambda\in\mathfrak{a}^*).$$

If $f\in C_c(G)$, then $\hat{f}_P(\sigma, \lambda)$ may make sense on $\hat{M}_d\times\mathfrak{a}_c^*$. Let $\phi\in\mathfrak{H}_\sigma$.

Then

$$(\hat{f}_P(\sigma, \lambda)\phi)(k_1)=\int_K \hat{f}_P(\sigma, \lambda; k_1, k_2)\phi(k_2)\,dk_2,$$

where

$$\hat{f}_P(\sigma, \lambda; k_1, k_2)=\int_{M\times A\times N} f(k_1 mank_2^{-1})e^{(\rho_P-i\lambda)(\log a)}\sigma(m)\,dmdadn.$$

We define the direct sum $\lambda\oplus\mu$ of $\lambda\in\mathfrak{a}_c^*$ and $\mu\in\mathfrak{a}_{Mc}^*$ as follows: If $H_0={}^*H+H$ ($H_0\in\mathfrak{a}_0$, ${}^*H\in\mathfrak{a}_M$, $H\in\mathfrak{a}$), then $(\lambda\oplus\mu)(H_0)=\mu({}^*H)+\lambda(H)$. Then $\lambda\oplus\mu\in\mathfrak{a}_{0c}^*$. By Lemma 1 we have that $\rho_0=\rho_P\oplus\rho_M$. Let M_0' be the normalizer of $\mathfrak{a}_0$ in K. We put $W=M_0'/M_0$, the Weyl group of G/K. We denote by $[W]$ the order of W. The group W acts on $\mathfrak{a}_0$ and on $\mathfrak{a}_{0c}^*$ by $sH=\mathrm{Ad}(k)H$ and $(s\lambda)(H)=\lambda(s^{-1}H)$ ($s=kM_0\in W$, $H\in\mathfrak{a}_0$ and $\lambda\in\mathfrak{a}_{0c}^*$). Let $\mathfrak{a}_0^+=\{H\in\mathfrak{a}_0 \mid \alpha(H)>0 \text{ for all } \alpha\in\Delta^+\}$, the positive Weyl chamber. We put $\mathfrak{a}_0'=\{H\in\mathfrak{a}_0 \mid \alpha(H)\neq 0 \text{ for all } \alpha\in\Delta^+\}$. Then for any connected component C of $\mathfrak{a}_0'$, we can take $s\in W$ uniquely so that $C=s\mathfrak{a}_0^+$. We put $A_0^+=\exp\mathfrak{a}_0^+$ and $A_0'=\exp\mathfrak{a}_0'$. Let C_{ρ_0} be the convex closure of $\{s\rho_0 \mid s\in W\}$ in $\mathfrak{a}_0^*$. We put

$$\mathcal{F}^1=\{\lambda\in\mathfrak{a}_c^* \mid \mathrm{Im}(\lambda\oplus\rho_M)\in C_{\rho_0}\},$$

where Im denotes the imaginary part.

Lemma 2. *Let f be a K-biinvariant and non-negative integrable function on G. If λ belongs to C_{ρ_0}, then we have*

$$\int_{A_0\times N_0} f(a_0n_0)\,e^{(\lambda+\rho_0)(\log a_0)}\,da_0dn_0\le [W]\|f\|_1,$$

where $\|f\|_1$ is the L^1 norm of f.

Proof. We write $a_0^s=\exp sH$ for $a_0=\exp H$. We put

$$F_f(a_0)=e^{\rho_0(\log a_0)}\int_{N_0} f(a_0n_0)\,dn_0.$$

Since f is K-biinvariant, we have $F_f(a_0^s)=F_f(a_0)$ for all $s\in W$ ([3, p. 261]). The measure of $A_0\backslash A_0'$ is zero and $A_0'=\bigcup_{s\in W}(A_0^+)^{s^{-1}}$ (disjoint union). Hence we have

$$\int_{A_0\times N_0} f(a_0n_0)e^{(\lambda+\rho_0)(\log a_0)}\,da_0dn_0=\sum_{s\in W}\int_{(A_0^+)^s} e^{\lambda(\log a_0)}F_f(a_0)\,da_0$$
$$\le\sum_{s\in W}\int_{A_0} e^{\rho_0(\log a_0)}F_f(a_0)\,da_0=[W]\|f\|_1. \qquad \text{Q.E.D.}$$

Let $f\in L^1(G)$. We put $f_1(x)=\int_{K\times K}|f(kxk')|\,dkdk$. Then f_1 is K-biinvariant and non-negative. Let us assume that $\lambda=\xi+i\eta\in\mathcal{F}^1$. Then,

$$\int_{K\times M\times A\times N\times K}|f(k_1mank_2^{-1})|\,e^{(\eta+\rho_P)(\log a)}\,dk_1dmdadndk_2$$
$$=\int_{M\times A\times N} f_1(man)\,e^{(\eta+\rho_P)(\log a)}\,dmdadn$$
$$=\int_{K_M\times A_M\times N_M\times A\times N} f_1({}^*k{}^*a{}^*nan)\,e^{2\rho_M(\log{}^*a)+(\eta+\rho_P)(\log a)}\,d{}^*k\,d{}^*a\,d{}^*n\,dadn$$
$$=\int_{A_0\times N_0} f_1(a_0n_0)\,e^{((\eta\oplus\rho_M)+\rho_0)(\log a_0)}\,da_0dn_0 \quad \text{(Lemma 1)}$$
$$\le[W]\|f_1\|_1 \quad \text{(Lemma 2)}=[W]\|f\|_1.$$

Therefore, by Fubini's theorem, the functions

$$\psi(m, a)=\int_N f(k_1 mank_2^{-1})dn \times e^{(\eta+\rho_P)(\log a)}$$

on $M \times A$ are integrable for almost all $(k_1, k_2) \in K \times K$. Hence if $\lambda=\xi+i\eta \in \mathcal{F}^1$, then $\hat{f}_P(\sigma, \lambda; k_1, k_2)$ for $f \in L^1(G)$ may be defined for almost all $(k_1, k_2) \in K \times K$. And $\hat{f}_P(\sigma, \lambda; k_1, k_2)$ is the Fourier transform of the integrable function ψ on the direct product group $M \times A$:

$$\hat{f}_P(\sigma, \lambda; k_1, k_2)=\int_{M\times A} \psi(m, a) e^{-i\xi(\log a)}\sigma(m)\, dmda.$$

Hence by [2, p. 317], if $(\sigma, \xi)\to\infty$ in the sense of the hull-kernel topology, then $\hat{f}_P(\sigma, \lambda; k_1, k_2)\to 0$. On the other hand, Lipsman's theorem ([5] and [7, p. 408]) says that the discrete series is discrete in hull-kernel topology. The space $\hat{M}_d$ is parametrized by a lattice in certain euclidean space and the unitary dual of a compact subgroup of M ([1]). Therefore, we have the same consequence as the above if $(\sigma, \xi)\to\infty$ in the topology of the parameter space. Thus we have the following

Theorem. *Let* $f \in L^1(G)$. *If* $(\sigma, \lambda) \in \hat{M}_d \times \mathcal{F}^1$, $\operatorname{Im} \lambda =$ *constant and* $(\sigma, \lambda)\to\infty$, *then* $\hat{f}_P(\sigma, \lambda; k_1, k_2)\to 0$ *for almost all* $(k_1, k_2) \in K \times K$.

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