105. An Extension of e^x to $[-\infty, \infty]$

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The sheaf $\mathcal{R}$ of Fourier hyperfunction on $[-\infty, \infty]$ is known to be a flabby sheaf. The exponential function e^x can be considered as a Fourier hyperfunctions on $[-\infty, \infty)$. Therefore, e^x can be extended to a Fourier hyperfunction on $[-\infty, \infty]$. In this paper, we will construct a concrete extension of e^x to $[-\infty, \infty]$.

§1. **Definitions.** We put $\boldsymbol{D}=[-\infty, \infty]$ and recall some definitions. $\tilde{\mathcal{O}}$ denotes the sheaf over $\boldsymbol{D}+i\boldsymbol{R}(i=\sqrt{-1})$ of slowly increasing holomorphic functions. For an open set Ω of $\boldsymbol{D}+i\boldsymbol{R}$, the section module $\tilde{\mathcal{O}}(\Omega)$ is the space of all holomorphic functions $f(z)\in\mathcal{O}(\Omega\cap\boldsymbol{C})$ such that for any $\varepsilon>0$ and any compact set K in Ω, the estimate

$$\sup\{|f(z)|e^{-\varepsilon|z|}\,;\, z\in K\cap\boldsymbol{C}\}<\infty$$

holds. For an open set ω of $\boldsymbol{D}$, the space $\mathcal{R}(\omega)$ of Fourier hyperfunctions on ω is defined to be

$$(1)\qquad \mathcal{R}(\omega)=\tilde{\mathcal{O}}(\Omega\backslash\omega)/\tilde{\mathcal{O}}(\Omega),$$

where Ω is a complex neighborhood of ω, i.e. Ω is an open set of $\boldsymbol{D}+i\boldsymbol{R}$, containing ω as a relatively closed subset. Let us remark that the right hand side of (1) is independent of the choice of complex neighborhoods Ω of ω. We mean by the flabbiness of the sheaf $\mathcal{R}$ the surjectivity of the restriction mappings

$$(2)\qquad \mathcal{R}(\omega_1)\to\mathcal{R}(\omega_2),$$

for any open subsets ω_1 and ω_2 of $\boldsymbol{D}$ such that $\omega_1\supset\omega_2$. For the details of the theory of Fourier hyperfunctions, we refer the reader to Sato [4] or Kawai [1].

The exponential function e^z belongs clearly to $\tilde{\mathcal{O}}([-\infty, \infty)+i\boldsymbol{R})$. We put

$$\exp_+(z)=\begin{cases} e^z & \text{for } \operatorname{Im} z>0\\ 0 & \text{for } \operatorname{Im} z<0.\end{cases}$$

Then $\exp_+(z)\in\tilde{\mathcal{O}}([-\infty, \infty)+i(\boldsymbol{R}\backslash 0))$ and the class $[\exp_+(z)]$ of $\exp_+(z)$ mod $\tilde{\mathcal{O}}([-\infty, \infty)+i\boldsymbol{R})$ is, by definition, the Fourier hyperfunction e^x on $[-\infty, \infty)$.

§2. **Fourier hyperfunction with support at $+\infty$.** For $M>0$, we put

$$(3)\qquad H_{M,\pi}=\{z=x+iy\in\boldsymbol{C}\,;\, x\geqslant M,\ |2xy|\leqslant\pi\}.$$

For $z\notin H_{M,\pi}$, we put

$$F(z)=\frac{1}{2\pi i}\int_{\partial H_{M,\pi}} \frac{\exp(-(z-w)^2)}{z-w}\exp(\exp w^2)dw. \tag{4}$$

By Cauchy's integral theorem, the right hand side of (4) is independent of $M>0$ and (4) defines an entire function $F(z)$ by analytic continuation.

For $a>0$ and $\varepsilon>0$, we denote $A=[a, \infty)$ and $A_\varepsilon[a-\varepsilon, \infty)+i[-\varepsilon, \varepsilon]$.

Proposition 1. *For any $R>0$, $a>0$, $\varepsilon>0$ and r with $0<r<1$, there exists $C\geqslant 0$ such that*

$$|F(z)|\leqslant C\exp(-rx^2) \quad \text{for } z=x+iy\notin A_\varepsilon \text{ and } |y|\leqslant R. \tag{5}$$

Proof. For $w=u+iv\in \boldsymbol{C}$, we have

$$|\exp(\exp w^2)|=\exp(\mathrm{Re}\exp w^2)=\exp(\exp(u^2-v^2)\cos 2uv)$$
$$\leqslant \exp(\exp(u^2-v^2))\leqslant \exp(\exp u^2).$$

If $|2uv|=\pi$ and $u\geqslant M>0$, we have

$$|\exp(\exp w^2)|=\exp(-\exp(u^2-v^2))$$
$$=\exp\left(-\exp\left(u^2-\left(\frac{\pi}{2u}\right)^2\right)\right)\leqslant \exp(-M'\exp u^2),$$

where $M'=\exp(-(\pi/2M)^2)>0$. Remark also

$$|\exp(-(z-w)^2)|=\exp(-(x-u)^2+(y-v)^2).$$

Let us fix $M>0$ so large that $A_\varepsilon\supset A_{\varepsilon/2}\supset H_{M,\pi}$. If $z=x+iy\notin A_\varepsilon$ and $|y|\leqslant R$, we have by (4)

$$|F(z)|\leqslant \frac{1}{2\pi}\int_{-\pi/2M}^{\pi/2M} C_0\exp(-(x-M)^2+\exp(M^2))dv \tag{6}$$
$$+\frac{1}{\pi}\int_M^\infty C_0\exp(-(x-u)^2-M'\exp(u^2))du,$$

where $C_0=2/\varepsilon\exp((R+\varepsilon)^2)$. Therefore, if we choose $C_1\geqslant 0$ sufficiently large, the first term of the right hand of (6) can be majorized by $C_1\exp(-rx^2)$. As for the second term, we can choose $C_2'\geqslant 0$ such that

The second term of the right hand of (6)

$$\leqslant C_2'\int_M^\infty \exp(-(x-u)^2-M'\exp(u^2))du$$
$$\leqslant C_2'\int_M^\infty \exp\left(-\left(\frac{x}{\sqrt{1+B}}-\sqrt{1+B}u\right)^2-\frac{B}{1+B}x^2\right)\exp(Bu^2-M'\exp(u^2))du$$
$$\leqslant C_2'\exp\left(-\frac{B}{1+B}x^2\right)\int_M^\infty \exp(Bu^2-M'\exp(u^2))du$$
$$=C_2''\exp\left(-\frac{B}{1+B}x^2\right),$$

where $B>0$ is an arbitrary positive number. Therefore, if we choose $C_2>0$ sufficiently large, the second term of the right hand of (6) can be majorized by $C_2\exp(-rx^2)$. Q.E.D.

By Proposition 1, the function $F(z)$ belongs to $\tilde{\mathcal{O}}((\boldsymbol{D}+i\boldsymbol{R})\backslash(+\infty+i0))$. The Fourier hyperfunction $T=[F]$, the class, of F mod $\tilde{\mathcal{O}}(\boldsymbol{D}+i\boldsymbol{R})$, has its support only at $\{+\infty\}$. It is proved in Morimoto-Yoshino [3] that T is not identically zero.

Remark. The function $F(z)$ never satisfies the estimate of the following type on A_ε:

(7) $|F(z)| \leqslant C \exp(B\, e^{\alpha|z|})$ for some $B>0$, $\alpha>0$ and $C \geqslant 0$.

If we have (7) for $z \in A_\varepsilon$, then F must belong to $\tilde{\mathcal{O}}(\boldsymbol{D}+i\boldsymbol{R})$ by the Phragmén-Lindelöf theorem and the Fourier hyperfunction $T=[F]$ must vanish identically. But this is not the case.

§ 3. Fourier transformations of T. Define the Fourier transformation of the Fourier hyperfunction T as follows:

$$\tilde{T}(\zeta)=\int_{\partial A_\varepsilon} e^{\zeta z} F(z)\, dz,$$

the integral being independent of $a>0$ and $\varepsilon>0$. $\tilde{T}(\zeta)$ is a non-zero entire function of exponential type in the following sense: For any $a_0 \geqslant 0$, $a>0$ and $\varepsilon>0$, there exists $C \geqslant 0$ such that

$$|\tilde{T}(\zeta)| \leqslant C \exp((-a+\varepsilon)|\operatorname{Re}\zeta| + \varepsilon|\operatorname{Im}\zeta|)$$

for every $\zeta \in \boldsymbol{C}$ with $\operatorname{Re}\zeta \leqslant a_0$. (See, for example, Morimoto [2].)

Restating Proposition 1, we get the following

Proposition 2. *Fix $\lambda>0$ such that $\tilde{T}(-\lambda) \neq 0$ and put $F_\lambda(z) = F(z/\lambda)/(\lambda \tilde{T}(-\lambda))$. Then F_λ is an entire function and for any $R>0$, $a>0$, $\varepsilon>0$ and r with $0<r<\lambda^{-2}$, there exists $C \geqslant 0$ such that*

$$|F_\lambda(z)| \leqslant C \exp(-rx^2) \quad \textit{for } z=x+iy \notin A_\varepsilon \textit{ and } |y| \leqslant R.$$

If we denote by $T_\lambda=[F_\lambda]$ the Fourier hyperfunction defined by F_λ, we have supp $T_\lambda=\{+\infty\}$.

§ 4. A differential equation. Let us consider the differential equation

(8) $$f'(z)-f(z)=F_\lambda(z)$$

in the complex plane $\boldsymbol{C}$. It is clear that a special solution f_+ is given as follows:

$$f_+(z)=e^z \int_{+\infty+iy}^{x+iy} e^{-w} F_\lambda(w)\, dw$$

for $y=\operatorname{Im} z>0$, where the integral path is a half straight line parallel to the x-axis. Similarly we define

$$f_-(z)=e^z \int_{+\infty+iy}^{x+iy} e^{-w} F_\lambda(w)\, dw$$

for $y=\operatorname{Im} z<0$.

Proposition 3. (i) *The functions f_+ and f_- can be extended to entire functions and satisfy the differential equation* (8).

(ii) *For any $R>0$, $\varepsilon>0$ and r $(0<r<\lambda^{-2})$, there exists $C \geqslant 0$ such that*

(9) $|f_+(z)| \leqslant C \exp(-rx^2)$ *for* $\varepsilon \leqslant \operatorname{Im} z \leqslant R$, $x=\operatorname{Re} z \geqslant 0$,

$|f_+(z)| \leqslant C \exp(-|x|)$ *for* $\varepsilon \leqslant \operatorname{Im} z \leqslant R$, $x=\operatorname{Re} z<0$;

(9′) $|f_-(z)| \leqslant C \exp(-rx^2)$ *for* $-\varepsilon \geqslant \operatorname{Im} z \geqslant -R$, $x=\operatorname{Re} z \geqslant 0$,

$|f_-(z)| \leqslant C \exp(-|x|)$ *for* $-\varepsilon \geqslant \operatorname{Im} z \geqslant -R$, $x=\operatorname{Re} z<0$.

(iii) $$f_+(z)-f_-(z)=e^z \qquad \text{for } z\in \boldsymbol{C}.$$

Proof. (i) As the function $e^{-w}F_\lambda(w)$ is an entire function, we can extend f_+ and f_- to the whole plane by analytic continuation.

(ii) Let us prove the estimate for f_+. Suppose $z=x+iy$ and Im $z>0$. We can rewrite the definition formula of f_+ as follows:

$$f_+(z)=e^z\int_{+\infty}^{x} e^{-(u+iy)}F_\lambda(u+iy)\,du.$$

By Proposition 2, if we choose r' with $0<r<r'<\lambda^{-2}$, we have

$$|f_+(z)|\leqslant e^x\left|\int_{+\infty}^{x} e^{-u}\,C\exp(-r'u^2)\,du\right|\leqslant C\,e^x\int_x^\infty \exp(-r'u^2-u)\,du.$$

If $x\geqslant 0$, we have

$$|f_+(z)|\leqslant C\exp(x-r'x^2)\int_0^\infty e^{-u}\,du=C\exp(x-r'x^2)\leqslant C'\exp(-rx^2).$$

If $x<0$, we have

$$|f_+(z)|\leqslant C\,e^x\int_x^\infty \exp(-r'u^2-u)\,du\leqslant C'e^x,$$

where C' is a constant.

(iii) By Cauchy's integral theorem, we have

$$\begin{aligned}f_+(z)-f_-(z)&=e^z\int_{\partial A_\varepsilon} e^{-w}F_\lambda(w)\,dw\\&=\frac{e^z}{\tilde{T}(-\lambda)}\int_{\partial A_\varepsilon} e^{-w}F(w/\lambda)\frac{dw}{\lambda}\\&=\frac{e^z}{\tilde{T}(-\lambda)}\int_{\partial A_\varepsilon(\lambda)} e^{-\lambda w}F(w)\,dw=e^z,\end{aligned}$$

where we put $A_\varepsilon(\lambda)=[a-\varepsilon/\lambda, \infty)+i[-\varepsilon/\lambda, \varepsilon/\lambda]$. Q.E.D.

Define the function $f_0(z)\in\tilde{\mathcal{O}}(\boldsymbol{D}+i(\boldsymbol{R}\backslash 0))$ as follows:

$$f_0(z)=\begin{cases}f_+(z) & \text{for Im } z>0\\ f_-(z) & \text{for Im } z<0.\end{cases}$$

Define the Fourier hyperfunction $E_\lambda=[f_0]$ on $\boldsymbol{D}$ as the class of f_0 mod $\tilde{\mathcal{O}}(\boldsymbol{D}+i\boldsymbol{R})$. Then by Proposition 3, (iii), the Fourier hyperfunction E_λ is an extension of the Fourier hyperfunction e^x on $[-\infty, \infty)$. Remark also the Fourier hyperfunction E_λ satisfies the following differential equation:

$$\frac{d}{dx}E_\lambda-E_\lambda=T_\lambda,$$

where T_λ is defined in §3 and satisfies supp $T_\lambda=\{+\infty\}$.

References

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