78. *On the Hessian of the Square of the Distance on a Manifold with a Pole*

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1980)

Analysis on a manifold with a pole has been studied in a series of papers by Greene-Wu. In particular, the characterization of C^n in terms of geometric conditions is one of the most interesting problems. In the case of a simply-connected complete Kähler manifold of non-positive curvature this problem has been solved by Siu-Yau [2] and Greene-Wu [1] (Theorem J). Concerning these results Wu has proposed some open problems in [4] and [5]. In this note we shall prove theorems related to his propositions. The author would like to thank Prof. Wu whose suggestion made this note materialize.

1. A smooth mapping $\phi: N \to M$ between Riemannian manifolds is called a *quasi-isometry* iff ϕ is a diffeomorphism and there exist positive constants μ and ν such that for each tangent vector X on N,

$$\mu |X|_N \leqq |\phi_*(X)|_M \leqq \nu |X|_N.$$

We recall that (M, o) is called a *manifold with a pole* iff M is a Riemannian manifold and the exponential mapping at $o \in M$ is a global diffeomorphism. Let (M, o) be a manifold with a pole. The distance function from the pole o will be denoted by r so that r^2 is a smooth function on M. The first theorem in question is the following

Theorem 1. *Let (M, o) be a manifold with a pole. Suppose there exists a continuous non-negative function $\varepsilon(t)$ on $[0, \infty)$ such that:*

(1) $|(1/2)D^2r^2 - g| \leqq \varepsilon(r)g,$

(2) $\varepsilon_o = \int_0^\infty (\varepsilon(t)/t)dt < \infty.$

Then $\exp: T_o(M) \to M$ *is a quasi-isometry satisfying*

$$\exp(\varepsilon_o)^{-1} |V| \leqq |\exp_*(V)| \leqq \exp(\varepsilon_o) |V|$$

for any tangent vector V at any point in $T_o(M)$.

In (1) above, D^2r^2 denotes the Hessian of the smooth function r^2 on M. Moreover inequality (1) means the following: If $x \in M$ and $X \in T_x(M)$ is a unit vector, then

$$\left| \frac{1}{2} D^2r^2(X, X) - 1 \right| \leqq \varepsilon(r(x)).$$

Remark. It follows from the above theorem that if (M, o) is a manifold with a pole and $(1/2)D^2r^2 = g$ on M then M is isometric to a Euclidian space. This is a weak form of a theorem by H. W. Wissner

[3]: If M is a connected complete Riemannian manifold and f is a smooth function on M whose Hessian is equal to the metric on M, then M is isometric to a Euclidian space.

Remark. Theorem C in [1] shows the following: Let (M, o) be a manifold with a pole. If there exist continuous functions $K, k: [0, \infty) \to [0, \infty)$ such that:

(1) $-k(r) \leqq$ radial curvature $\leqq K(r)$,

(2) $\int_0^\infty tK(t)dt \leqq 1$,

(3) $\int_0^\infty tk(t)dt < \infty$,

then $\exp: T_o(M) \to M$ is a quasi-isometry. On the other hand Theorm in [5] says that under the same assumption as in Theorem C in [1] there exists a positive smooth function $\varepsilon(t)$ on $[0, \infty)$ such that $\varepsilon(t) \to 0$ as $t \to \infty$ and the conditions (1) and (2) in Theorem 1 above are satisfied. Therefore Theorem C in [1] follows from Theorem in [5] and Theorem 1.

2. Let (M, o) be a manifold with a pole and r the distance function from o and ∂ the radial vector field, so that ∂ is the gradient of r. We define a vector field H by $H = r\partial$. Then a straight calculation shows that

$$H = \frac{1}{2} \operatorname{grad}(r^2),$$

and particularly H is a smooth vector field on M. We denote by (ϕ_t) the one parameter transformation group of M generated by the vector field H. We have the following

Lemma 1. *If the Lie derivative by the vector field H is denoted by $\mathcal{L}_H$, then we have*

$$D^2 r^2 = \mathcal{L}_H g.$$

Proof. Let X and Y be vector fields on M. Since $H = 1/2 \operatorname{grad}(r^2)$, $D^2 r^2(X, Y) = X(Y(r^2)) - \nabla_X Y(r^2) = X(\operatorname{grad}(r^2), Y) - (\operatorname{grad}(r^2), \nabla_X Y) = 2(\nabla_X H, Y)$. On the other hand the torsion of the Riemannian connection ∇ is free, thus $\mathcal{L}_H g(X, Y) = H(X, Y) - ([H, X], Y) - (X, [H, Y]) = (\nabla_X H, Y) + (X, \nabla_Y H)$. Since $D^2 r^2$ is symmetric, $(\nabla_X H, Y) = (1/2) D^2 r^2(X, Y) = (1/2) D^2 r^2(Y, X) = (\nabla_Y H, X)$. Hence $D^2 r^2(X, Y) = \mathcal{L}_H g$.

Lemma 2. *For any vector v in $T_o(M)$, we have*

$$\phi_t(\exp v) = \exp(e^t v).$$

Proof. Let v be a unit vector in $T_o(M)$ and $\gamma(s) = \exp(sv)$. Then $(d/dt)(\phi_t(\gamma(s)))_t = H_{\phi_t(\gamma(s))} = r(\phi_t(\gamma(s)))\partial_{\phi_t(\gamma(s))}$. Now we define $\Phi(0, s)$ by the following: $\phi_t(\gamma(s)) = \gamma(\Phi(t, s))$, so that $r(\phi_t(\gamma(s))) = \Phi(t, s)$ and $\Phi(t, s) = s$. Then $(d/dt)(\phi_t(\gamma(s)))_t = ((\partial/\partial t)\Phi(t, s))_t \partial_{\gamma(\Phi(t,s))}$ and hence $(\partial/\partial t)\Phi(t, s) = \Phi(t, s)$. Since we have $\Phi(0, s) = s$, $\Phi(t, s) = se^t$. Therefore $\phi_t(\gamma(s)) = \gamma(se^t)$, i.e., $\phi_t(\exp sv) = \exp(e^t sv)$.

Let $\gamma=\{\exp tv : t\geqq 0\}$ be a ray from o ($v\in T_o(M)$, $|v|=1$) and V a vector in $T_v(T_o(M))$ orthogonal to v. Regarding sV a vector in $T_{sv}(T_o(M))$ in the usual way, we define a vector field along γ by

$$Z_{\exp(sv)}=(\exp_*)_{sv}(sV).$$

Clearly Z is a Jacobi field along the ray γ and Z is orthogonal to γ at every point. Furthermore any Jacobi field along γ which vanishes at o and is orthogonal to γ can be obtained in this way. The Jacobi field Z is called *the Jacobi field along γ defined by V.*

Lemma 3. *Let Z be the Jacobi field along γ defined by V and $f(r(x))=|Z_x|$ ($x\in\gamma$). Then we have*

(1) *Z is (ϕ_t)-invariant and $[H,Z]=0$,*

(2) $\lim_{r\to 0} f(r)=0$ *and* $\lim_{r\to 0} f(r)/r=|V|$,

(3) $(1/2)\mathcal{L}_H g(Z,Z)=rf(r)f'(r)$.

Proof. For any $u\in T_o(M)$, $\phi_t(\exp u)=\exp(e^t u)$. Hence $(\phi_t)_*(Z_{\exp(sv)})=(\phi_t)_*((\exp_*)_{sv}(sV))=(\exp_*)_{e^t sv}(e^t sV)=Z_{\exp(e^t sv)}=Z_{\phi_t(\exp(sv))}$ and then Z is (ϕ_t)-invariant. This shows (1). The first limit of (2) is obvious. $f(r)/r=|(\exp_*)_{rv}(rV)|/r=|V|\,(|\exp_*)_{rv}(rV)|/r\,|V|)\to|V|$ as $r\to 0$, this shows the second limit of (2). Since we have $[H,Z]=0$, (3) can be obtained as follows;

$$\frac{1}{2}\mathcal{L}_H g(Z,Z)=\frac{1}{2}H(|Z|^2)-([H,Z],Z)=\frac{1}{2}r\partial(|Z|^2)$$
$$=\frac{1}{2}r(f(r)^2)'=rf(r)f'(r).$$

3. Proof of Theorem 1. Let (M,o) be a manifold with a pole which satisfies conditions in Theorem 1. Namely, we have a non-negative continuous function $\varepsilon(t)$ on $[0,\infty)$ satisfying the conditions (1) and (2). Let Z be a Jacobi field along a ray γ defined by V and $f(r(x))=|Z_x|$ ($x\in\gamma$). Then the condition (1) in Theorem 1 implies $|rf(r)f'(r)-f(r)^2|\leqq\varepsilon(r)f(r)^2$ since $(1/2)\mathcal{L}_H g(Z,Z)=rf(r)f'(r)$. Therefore we have

$$-\frac{\varepsilon(r)}{r}\leqq\frac{rf'(r)-f(r)}{f(r)r}\leqq\frac{\varepsilon(r)}{r}.$$

Since the mid-term equals $(f(r)/r)'/(f(r)/r)$, we have

$$-\int_0^r \varepsilon(t)/t\,dt\leqq\log f(t)|_0^r\leqq\int_0^r \varepsilon(t)/t\,dt.$$

Since

$$\lim_{r\to 0} f(r)/r=|V| \quad\text{and}\quad \varepsilon_o=\int_0^\infty \varepsilon(t)/t\,dt,$$
$$\log|V|-\varepsilon_o\leqq\log f(r)/r\leqq\log|V|+\varepsilon_o,$$

and hence

$$|V|\exp(-\varepsilon_o)\leqq f(r)/r\leqq|V|\exp(\varepsilon_o),$$

that is,

$$|rV|\exp(-\varepsilon_o)\leqq|(\exp_*)_{rv}(rV)|\leqq|rV|\exp(\varepsilon_o).$$

Therefore for any w in $T_o(M)$ and W in $T_w(T_o(M))$ orthogonal to w,

$$|W| \exp(-\varepsilon_o) \leqq |(\exp_*)_w(W)| \leqq |W| \exp(\varepsilon_o).$$

On the other hand, the restriction of exp to the ray γ is an isometry. Hence for any $v \in M$ and any $V \in T_v(T_o(M))$, the same inequality holds. This completes the proof.

Remark. If (M, o) is a manifold with a pole, then there exists a non-negative continuous function $\varepsilon(t)$ on $[0, \infty)$ such that:

(1) $|(1/2)D^2r^2 - g| \leqq \varepsilon(r)g$ around o.

(2) $\varepsilon(t)/t$ is bounded around $t=0$.

Therefore we can prove the following: If there exists a non-negative continuous function $\varepsilon(t)$ satisfying (1) in Theorem 1 and

$$\int_c^\infty \varepsilon(t)/t\, dt < \infty \qquad \text{for some } c>0,$$

then $\exp: T_o(M) \to M$ is a quasi-isometry.

4. A manifold (M, o) with a pole is called a *model* iff the linear isotropy group of isometries at o is the full orthogonal group. If (M, o) is a model then the metric g of (M, o) relative to geodesic polar coordinates centered at o assumes the form

$$g = dr^2 + f(r)^2 d\theta^2,$$

where f is a smooth function on $[0, \infty)$ satisfying

(1) $f>0$ on $[0, \infty)$

(2) $f(0)=0$, $f'(0)=1$.

In this case the radial curvature κ becomes a function of distance function r and is called the radial curvature function. Then the Jacobi equation is

$$f''(t) = -\kappa(t) f(t).$$

We have the following propositions on a model with respect to the conditions of Theorem 1.

Proposition 1. *Let (M, o) be a model with a non-positive radial curvature function $-k$, i.e., $k \geqq 0$. We define the function $\varepsilon: [0, \infty) \to \boldsymbol{R}$ by $\varepsilon(t) = (1/2)D^2r^2(X, X) - 1$, where $X \in T_x(M)$ with $r(x)=t$, $|X|=1$ and X is orthogonal to ∂_x. Then $\varepsilon(t) \geqq 0$ and the following conditions are equivalent:*

(A) $\exp: T_o(M) \to M$ *is a quasi-isometry.*

(B) *There is some constant $\eta \geqq 1$ such that $r \leqq f(r) \leqq \eta r$.*

(C) *There is some constant $\eta \geqq 1$ such that $1 \leqq f'(r) \leqq \eta$.*

(D) $\int_0^\infty s k(s) ds < \infty$.

(E) $\int_0^\infty \varepsilon(s)/s\, ds < \infty$.

The equivalence of the first four conditions was proved by Greene-Wu [1] (Lemma 4.5), the implication of (D) to (E) was obtained by Wu [5] and Theorem 1 of this note says that (E) implies (A). Hence all conditions are equivalent.

Similarly we can show the following proposition for the case of non-negative curvature.

Proposition 2. *Let (M, o) be a model with a non-negative curvature function K, i.e., $K \geqq 0$. We define the function $\varepsilon : [0, \infty) \to \boldsymbol{R}$ by $\varepsilon(t) = 1 - (1/2) D^2 r^2(X, X)$, where $X \in T_x(M)$ with $r(x) = t$, $|X| = 1$ and X is orthogonal to ∂_x. Then $\varepsilon(t) \geqq 0$ and the following conditions are equivalent:*

(A) $\exp : T_o(M) \to M$ *is a quasi-isometry.*

(B) *There is a constant η, $0 < \eta \leqq 1$, such that $\eta r \leqq f(r) \leqq r$.*

(C) *There is a constant η, $0 < \eta \leqq 1$, such that $\eta \leqq f'(r) \leqq 1$.*

(D) $\int_0^\infty sk(s) ds \leqq 1$.

(E) $\int_0^\infty \varepsilon(s)/s \, ds < \infty$.

5. We shall show the second theorem of this note which resembles the converse when the radial curvature is non-positive. Let v be a unit vector in $T_o(M)$ and V a vector in $T_v(T_o(M))$ orthogonal to v and Z the Jacobi field along $\gamma = \{\exp tv (t \geqq 0)\}$ defined by V. We define $\kappa_{\gamma,Z}(t)$ and $\varepsilon_{\gamma,Z}(t)$ as follows:

$\kappa_{\gamma,Z}(t)$ = the radial curvature of the plane spanned by ∂ and Z at $\exp tv$,

$$\varepsilon_{\gamma,Z}(t) = \frac{1}{|Z|^2}\left(\frac{1}{2} D^2 r^2(Z, Z) - |Z|^2\right) \qquad \text{at } \exp tv.$$

Using this notation we shall prove the following

Theorem 2. *Let (M, o) be a manifold with a pole whose radial curvature is non-positive. Suppose $\exp : T_o(M) \to M$ is a quasi-isometry. Then for any Jacobi field Z along a ray γ defined by V,*

(1) $0 \leqq \varepsilon_{\gamma,Z}(t)$,

(2) $\int_0^\infty \varepsilon_{\gamma,Z}(t)/t \, dt < \infty$,

(3) $0 \leqq \int_0^\infty -\kappa_{\gamma,Z}(t) t \, dt < \infty$.

Proof. Since the radial curvature is non-positive, we have that

$$|V| \leqq |\exp_* (V)| \qquad \text{for any } V \in T_v(T_o(M)) \ (v \in T_o(M)).$$

Therefore there exists a positive constant $\eta \geqq 1$ such that

$$|V| \leqq |\exp_* (V)| \leqq \eta |V| \qquad \text{for any } V \in T_v(T_o(M)) \ (v \in T_o(M)).$$

Let $f(r(x)) = |Z_x|$ $(x \in \gamma)$. Thus if $x = \phi_t(\exp v)$, then $r(x) = r(\phi_t(\exp v)) = r(\exp e^t v) = e^t$. Hence $f(r(x)) = |Z_x| = |(\exp_*)_{e^t v}(e^t v)|$ and so $|e^t V| \leqq f(r(x)) \leqq \eta |e^t V|$, that is,

(4) $r|V| \leqq f(r) \leqq \eta r |V|$.

On the other hand, since Z is a Jacobi field, Z satisfies the Jacobi equation along γ, that is,

$$\nabla_\partial^2 Z + R(Z, \partial)\partial = 0 \qquad \text{along } \gamma.$$

Thus $0=(\nabla_\partial^2 Z, Z)+\kappa_{\gamma,Z}(r)|Z|^2$. Moreover we have $(\nabla_\partial^2 Z, Z)=\partial(\nabla_\partial Z, Z)-|\nabla_\partial Z|^2=(1/2)(f(r)^2)''-|\nabla_\partial Z|^2$. Since the parallel displacement by ∇ is an isometry, $|\nabla_\partial Z|\geqq|\partial|Z||=|f'(r)|$. Therefore we have $(1/2)(f(r)^2)''-(f'(r))^2\geqq-\kappa_{\gamma,Z}(r)f(r)^2$, and hence

(5) $f''(r)\geqq-\kappa_{\gamma,Z}(r)f(r)$.

Since $\kappa_{\gamma,Z}(r)\leqq 0$, $f(r)$ is an increasing convex function. Moreover we claim

(6) $|V|\leqq f'(r)\leqq\eta|V|$.

In fact, $f'(0)=|V|$ by Lemma 3 (2). Thus $|V|\leqq f'(r)$. Suppose there exists $r_o\geqq 0$ such that $f'(r_o)>\eta|V|$. Take a small positive $\varepsilon>0$ such that $f'(r_o)>\eta|V|+\varepsilon$. Since $f'(r)$ is an increasing function, $f'(r)>\eta|V|+\varepsilon\,(r\geqq r_o)$. Thus $f(r)-f(r_o)>(\eta|V|+\varepsilon)(r-r_o)$. If r is sufficiently large, inequality (4) is contradicted. Therefore we have the inequality (6). Now we can show the inequality (1) as follows:

$$\varepsilon_{\gamma,Z}(r)=\frac{1}{|Z|^2}\left(\frac{1}{2}D^2r^2(Z,Z)-|Z|^2\right)=\frac{1}{|Z|}H(|Z|)-1$$
$$=\frac{1}{f(r)}(rf'(r)-f(r))\geqq r(f(r)/r)'/(f(r)/r).$$

Since $f(r)$ is an increasing convex function and $f(0)=0$, $f(r)/r$ is an increasing function and hence $\varepsilon(r)\geqq 0$. Thus we have

$$\int_0^\infty \varepsilon_{\gamma,Z}(t)/t\,dt=\log f(r)/r|_0^\infty\leqq\log\eta|V|-\log|V|=\log\eta<\infty.$$

This shows (2). By integrating the inequality (5) we have $f'(r)-f'(0)\geqq\int_0^r-\kappa_{\gamma,Z}(t)f(t)dt$. Since $f'(0)=|V|$ and $f(r)\geqq r|V|$, $\int_0^r-\kappa_{\gamma,Z}(t)t\,dt\leqq\int_0^r-\kappa_{\gamma,Z}(t)(f(t)/|V|)dt\leqq(1/|V|)(f'(r)-|V|)\leqq\eta-1<\infty$. The proof is completed.

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