75. *Remarks on the Lower Bound of a Linear Operator*

By Gyokai NIKAIDO
Department of Mathematics, Science University of Tokyo

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1. Introduction. Let X, Y be normed linear spaces and let T be a linear operator with domain $\mathrm{D}(T)$ in X and range $\mathrm{R}(T)$ in Y. By $\mathrm{N}(T)$, we denote the null space of T and set $n(T)=\dim \mathrm{N}(T)$. The set of all closed linear operators from X to Y is denoted by $\mathcal{C}(X, Y)$.

The *lower bound* (or *reduced minimum modulus*), $\gamma(T)$, of T is defined by

$$\gamma(T)=\sup \{\gamma : \|Tx\| \geqq \gamma d(x, \mathrm{N}(T)) \ (x \in \mathrm{D}(T))\}$$

where $d(x, \mathrm{N}(T))$ denotes the distance from x to $\mathrm{N}(T)$. If X, Y are Banach spaces and $T \in \mathcal{C}(X, Y)$, then $\mathrm{R}(T)$ is closed if and only if $\gamma(T) > 0$ (cf. Kato [1, p. 231]). A closed linear operator with closed range is called *normally solvable*.

Now let Z be another normed linear space and let S be a linear operator from Y to Z. Then the following result is well known.

Theorem 1. *Assume that*

(1) *X, Y and Z are Banach spaces;*

(2) *$T \in \mathcal{C}(X, Y)$ and $S \in \mathcal{C}(Y, Z)$ are normally solvable;*

(3) $n(S) < \infty$.

Then ST is also normally solvable.

For the proof of the above theorem, we refer to Kato [2, p. 277].

In this note, we are interested in the estimate of $\gamma(ST)$ from below in terms of $\gamma(S)$ and $\gamma(T)$. As a result, we shall obtain Theorem 1 above as a corollary.

2. Estimate of $\gamma(ST)$. Before we state our result, we shall explain some notations. Let E be a normed linear space and let M, N be closed subspaces of E. For such a pair (M, N), we define the quantity $\gamma(M, N)$ by

$$\gamma(M, N)=\inf \frac{d(u, N)}{d(u, M \cap N)}$$

where infimum is taken over all u such that $u \in M$ and $u \notin N$. If $M \subset N$, we set $\gamma(M, N)=1$. For a Banach space E, it is known that $\gamma(M, N) > 0$ if and only if $M+N$ is closed in E. For details, we refer to Kato [1].

Let $x \in \mathrm{D}(T)$. Then we have the following

Lemma 1. $d(Tx, \mathrm{N}(S)) \geqq \gamma(T)\gamma(\mathrm{R}(T), \mathrm{N}(S)) d(x, \mathrm{N}(ST))$.

Proof. Since we have

$$d(Tx, \mathrm{N}(S)) \geqq \gamma(\mathrm{R}(T), \mathrm{N}(S)) d(Tx, \mathrm{N}(S) \cap \mathrm{R}(T)),$$

it suffices to prove that

$$d(Tx, \mathrm{N}(S) \cap \mathrm{R}(T)) \geqq \gamma(T) d(x, \mathrm{N}(ST)).$$

It follows from $T\mathrm{N}(ST) = \mathrm{N}(S) \cap \mathrm{R}(T)$ that

$$\begin{aligned} d(Tx, \mathrm{N}(S) \cap \mathrm{R}(T) &= \inf \{\|T(x-z)\| : z \in \mathrm{N}(ST)\} \\ &\geqq \gamma(T) \inf \{d(x-z, \mathrm{N}(T)) : z \in \mathrm{N}(ST)\} \\ &\geqq \gamma(T) d(x, \mathrm{N}(ST)). \end{aligned}$$

This completes the proof of the lemma.

By using Lemma 1, we now obtain the estimate of $\gamma(ST)$ from below in terms of $\gamma(S)$ and $\gamma(T)$.

Proposition 1. $\gamma(ST) \geqq \gamma(S)\gamma(T)\gamma(\mathrm{R}(T), \mathrm{N}(S))$.

Proof. Let $x \in \mathrm{D}(ST)$. Then it follows from Lemma 1 that

$$\begin{aligned} \|STx\| &\geqq \gamma(S) d(Tx, \mathrm{N}(S)) \\ &\geqq \gamma(S)\gamma(T)\gamma(\mathrm{R}(T), \mathrm{N}(S)) d(x, \mathrm{N}(ST)), \end{aligned}$$

whence we have

$$\gamma(ST) \geqq \gamma(S)\gamma(T)\gamma(\mathrm{R}(T), \mathrm{N}(S)).$$

3. Corollaries of Proposition 1. In this section, we state some corollaries of Proposition 1. We shall assume throughout that X, Y, Z are Banach spaces and $T \in \mathcal{C}(X, Y)$, $S \in \mathcal{C}(Y, Z)$ are both normally solvable.

Corollary 1. *Let T, S be bounded with $\mathrm{D}(T) = X$ and $\mathrm{D}(S) = Y$. Then ST is normally solvable if and only if $\mathrm{N}(S) + \mathrm{R}(T)$ is closed in Y.*

Proof. Assume that ST is normally solvable. Then $\mathrm{R}(ST)$ is closed, so that $\mathrm{N}(S) + \mathrm{R}(T) = S^{-1}\mathrm{R}(ST)$ is closed in Y since S is bounded with $\mathrm{D}(S) = Y$. The converse is a direct consequence of Proposition 1.

Corollary 2. *Assume that $n(S) < \infty$. Then ST is normally solvable.*

Proof. Since $ST \in \mathcal{C}(X, Z)$ (cf. Kato [2, p. 277]), it suffices to note that $\gamma(\mathrm{R}(T), \mathrm{N}(S)) > 0$.

Finally, we shall consider a bounded normal operator T on a Hilbert space H. Then it is easy to verify that $\overline{\mathrm{R}(T)} = \mathrm{N}(T)^{\perp}$ and $\mathrm{N}(T^n) = \mathrm{N}(T)$ for every positive integer n, where $\overline{\mathrm{R}(T)}$ denotes the closure of $\mathrm{R}(T)$ and $\mathrm{N}(T)^{\perp}$ denotes the orthogonal complement of $\mathrm{N}(T)$.

Corollary 3. *Assume that H is a Hilbert space and T is a normal operator on H. Then we have;*

$$\gamma(T^n) \geqq [\gamma(T)]^n \qquad (n = 2, 3, \cdots).$$

Proof. Let $n \geqq 2$. Then it follows from Proposition 1 that

$$\gamma(T^n) \geqq (\gamma_1 \gamma_2 \cdots \gamma_{n-1})[\gamma(T)]^n,$$

where $\gamma_k = \gamma(\mathrm{R}(T), \mathrm{N}(T^k))$ $(k = 1, 2, \cdots, n-1)$. Hence it is enough to show that $\gamma_k = 1$ for $k = 1, 2, \cdots, n-1$. However, by the remark preceding the corollary, we have

$$\gamma_k = \gamma(\mathrm{R}(T), \mathrm{N}(T^k)) = \gamma(\mathrm{R}(T), \mathrm{N}(T)) = 1,$$

which completes the proof.

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References

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[2] ——: Perturbation theory for nullity, deficiency and other quantities of linear operators. J. Anal. Math., **6**, 273–322 (1958).