

53. *Integration of Correspondences and a Variational Problem with Operator Constraint**

By Toru MARUYAMA

Department of Economics, Keio University

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1980)

1. **Introduction.** Let $u: [0, 1]^2 \times R_+^l \rightarrow R$ and $\omega: [0, 1] \rightarrow R_+^l$ be two fixed mappings, and consider the following variational problem with a control variable $x: [0, 1]^2 \rightarrow R_+^l$:

$$\begin{aligned}
 (*) \quad & \underset{x}{\text{Maximize}} \int_0^1 \int_0^1 u(s, t, x(s, t)) ds dt \\
 & \text{subject to} \\
 & \int_0^1 x(s, t) ds \leq \omega(t) \quad \text{for all } t \in [0, 1].
 \end{aligned}$$

(R_+^l designates the nonnegative orthant of R^l .)

We can easily give this problem a lot of economic interpretations. For example, $x(s, t)$ can be interpreted as an allocation of various resources among agents $s \in [0, 1]$ over time-interval $[0, 1]$. The available quantities of these resources at each time $t \in T$ are represented by $\omega(t)$. Then (*) is a formal expression of the problem to maximize the sum of utilities of all agents over time subject to the resource-constraint. (Cf. Maruyama [10] for a related problem.)

In this paper, we are going to establish a set of sufficient conditions which assures the existence of an optimal solution for this kind of variational problem. Several new results on infinite dimensional Ljapunov measures and the integration theory of correspondences (= multivalued mappings) will also be presented as indispensable preparations for our main purpose.

Arkin-Levin [1] examined a similar problem and I am very much indebted to them for various ideas.

2. **Abstract integrals of correspondences.** Throughout this section, Ω stands for a compact Hausdorff space, \mathfrak{X} and \mathfrak{Y} for Banach spaces, and $C(\Omega, \mathfrak{X})$ for the set of all the continuous mappings $f: \Omega \rightarrow \mathfrak{X}$. $C(\Omega, \mathfrak{X})$ is a Banach space whose topology is defined by the sup-norm:

$$\|f\| = \sup_{\omega \in \Omega} \|f(\omega)\|.$$

We designate by $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ the space of all the bounded linear operators on \mathfrak{X} into \mathfrak{Y} . Let

*) The earlier version of this paper was read at the annual meeting of the Japan Association of Economics and Econometrics in 1979. The financial support from Keio Gijuku Academic Promotion Funds is gratefully acknowledged.

$$T: \mathcal{C}(\Omega, \mathfrak{X}) \rightarrow \mathfrak{Y}$$

be any bounded linear operator. Then T is *dominated*; i.e. there exists a positive regular Borel measure γ on Ω such that

$$(1) \quad \|Tf\| \leq \int_{\Omega} \|f(\omega)\| d\gamma \quad \text{for all } f \in \mathcal{C}(\Omega, \mathfrak{X}).$$

Accordingly, there exists uniquely an $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ -valued regular Borel measure τ with finite variation $|\tau|$ such that

$$(2) \quad Tf = \int_{\Omega} f d\tau \quad \text{for all } f \in \mathcal{C}(\Omega, \mathfrak{X}),$$

and the least regular Borel measure which satisfies (1) is identical with $|\tau|$. Since $\mathcal{C}(\Omega, \mathfrak{X})$ is dense in

$$L^1_{\mathfrak{X}}(|\tau|) = \left\{ f: \Omega \rightarrow \mathfrak{X} \mid \int_{\Omega} \|f(\omega)\| d|\tau| < +\infty \right\},$$

the operator T corresponding to τ can be extended as a bounded linear operator on $L^1_{\mathfrak{X}}(|\tau|)$ into \mathfrak{Y} . For the detailed discussion about these concepts and results, consult Dinculeanu [6, § 19].

Proposition 1 (Diestel-Uhl [5, pp. 101–102]). *Let (X, \mathcal{E}, m) be a finite measure space and \mathfrak{X} be a Banach space such that \mathfrak{X} and \mathfrak{X}' have the Radon-Nikodým property. A subset $K \subset L^1_{\mathfrak{X}}(m)$ is relatively weakly compact if*

- (i) K is bounded,
- (ii) K is uniformly integrable, and
- (iii) for every $E \in \mathcal{E}$, the set

$$\left\{ \int_E f dm \mid f \in K \right\}$$

is relatively weakly compact.

Using this proposition, we can get the following theorem as a generalization of Castaing [2], where Γ is assumed to be constant.

Theorem 1. *Let \mathfrak{X} be a Hilbert space, $\Gamma: \Omega \twoheadrightarrow \mathfrak{X}$ be a compact-convex-valued measurable correspondence, and \mathcal{F}_{Γ} be the set of all measurable selections of Γ . If Γ is integrably bounded; i.e. there exists some $\psi \in L^1(|\tau|)$ such that*

$$\sup_{f \in \mathcal{F}_{\Gamma}} \|f(\omega)\| \leq \psi(\omega) \text{ a.e.,}$$

then \mathcal{F}_{Γ} is weakly compact and convex in $L^1_{\mathfrak{X}}(|\tau|)$.

The following theorem is an easy consequence of Theorem 1.

Theorem 2. *Under the assumptions in Theorem 1, $T(\mathcal{F}_{\Gamma})$ is weakly compact and convex in \mathfrak{Y} .*

3. A variational problem with operator constraint. Let us begin by specifying some notations!

S, T : compact metric spaces.

μ : non-atomic positive Radon measure on S .

ν : non-atomic positive Radon measure on T .

$\Gamma: S \times T \twoheadrightarrow \mathbf{R}^1$: compact-valued measurable correspondence.

compact-convex-valued, $H(\mathcal{F}_\Delta)$ is weakly compact by Theorem 2, provided that Δ is integrably bounded. (A simple sufficient condition for Δ to be integrably bounded is that the functions

$$(7) \quad \begin{aligned} \bar{u}(s, t) &= \sup_{x \in \Gamma(s, t)} |u(s, t, x)| < +\infty \\ \bar{g}(s, t) &= \sup_{x \in \Gamma(s, t)} \|g(s, t, x)\| < +\infty \end{aligned}$$

are summable with respect to $\mu \otimes \nu$.)

Since $H(\mathcal{F}_\Delta)$ is weakly compact,

$$K \equiv \{(a, b(t)) \in H(\mathcal{F}_\Delta) \mid b(t) \leq \omega(t)\}$$

is also weakly compact. Therefore the projection of K into R is compact and hence has the greatest element a^* . Pick up any $b^*(t) \in L^1_{R^i}(\nu)$ such that $(a^*, b^*(t)) \in K$.

If we can prove the relation

$$(8) \quad H(\mathcal{F}_\Delta) = H(\mathcal{F}_{\Delta^*}),$$

then there exists a measurable selection $\lambda^*(s, t)$ of $\Delta(s, t)$ which satisfies $H(\lambda^*) = (a^*, b^*)$. Since the set

$$(9) \quad \{(s, t, x) \in S \times T \times R^l \mid f(s, t, x) = \lambda^*(s, t), x \in \Gamma(s, t)\}$$

is clearly measurable in $S \times T \times R^l$, there exists a measurable selection $x^*(s, t)$ of $\Gamma(s, t)$ such that

$$(10) \quad [u(s, t, x^*(s, t)), g(s, t, x^*(s, t))] = \lambda^*(s, t).$$

Hence we can conclude that x^* is clearly an optimal solution of the problem (P).

Thus, it remains only to establish (8).

3° Let (X, \mathcal{E}) be a measurable space and \mathfrak{X} be a Banach space. If $m: \mathcal{E} \rightarrow \mathfrak{X}$ is a countably additive \mathfrak{X} -valued measure, then there exists a nonnegative countably additive (scalar-valued) measure γ such that

$$\gamma(E) = 0 \Leftrightarrow m(E \cap F) = 0 \quad \text{for all } F \in \mathcal{E}.$$

Using these notations, we can summarize the result of G. Knowles.

Proposition 2 (Knowles [8]). *The following two statements are equivalent.*

(i) m is Ljapunov.

(ii) If $\gamma(E) > 0$, then the mapping $f \mapsto \int_E f dm$ is not injective on $L^\infty(E) = \{f \text{ is } \gamma\text{-essentially bounded and } f(x) = 0 \text{ on } E^c\}$.

As an application of this proposition, we can prove the next important theorem.

Theorem 3. τ is Ljapunov.

Since the correspondence $\Delta: S \times T \twoheadrightarrow R^{l+1}$ is compact-convex-valued, the profile $(\Delta(s, t))''$ of $\Delta(s, t)$ is not empty for all $(s, t) \in S \times T$. If we define the correspondence $\dot{\Delta}: S \times T \twoheadrightarrow R^{l+1}$ by

$$\dot{\Delta}: (s, t) \twoheadrightarrow (\Delta(s, t))'',$$

then $\dot{\Delta}$ is also measurable. On the other hand, if Δ is integrably bounded, then \mathcal{F}_Δ is weakly compact and convex by Theorem 1. Hence

$\ddot{\mathcal{F}}_A \neq \phi$. In fact, we can prove the simple relation $\ddot{\mathcal{F}}_A = \mathcal{F}_A$ (cf. Castaing [3], Maruyama [11, § 5]). Furthermore, combining Theorem 2.6 of [1] and our Proposition 2, we can get the following crucial result.

Theorem 4. *If Δ is integrably bounded, then*

$$\int_{S \times T} \Delta d\tau = \int_{S \times T} \dot{\Delta} d\tau.$$

Corollary 1. *Let $f_i \in L^1(\mu \otimes \nu)$, $g_i \in L^1_{R^i}(\mu \otimes \nu)$ ($i=1, 2, \dots, p$). Then for any measurable mapping*

$$\sigma : S \times T \rightarrow \Sigma_p \equiv \left\{ (\sigma_1, \sigma_2, \dots, \sigma_p) \in R^p_+ \mid \sum_{i=1}^p \sigma_i = 1 \right\},$$

there exist pairwise disjoint measurable sets $M_i \subset S \times T$ ($i=1, 2, \dots, p$) such that

$$\begin{aligned} S \times T &= \bigcup_{i=1}^p M_i \\ \int_{S \times T} \sum_{i=1}^p \sigma_i(s, t) f_i(s, t) d(\mu \otimes \nu) &= \int_{S \times T} \sum_{i=1}^p f_i(s, t) \chi_{M_i}(s, t) d(\mu \otimes \nu) \\ \int_S \sum_{i=1}^p \sigma_i(s, t) g_i(s, t) d\mu &= \int_S \sum_{i=1}^p g_i(s, t) \chi_{M_i}(s, t) d\mu. \end{aligned}$$

Let $(\alpha(s, t), \beta(s, t)) \in L^1(\mu \otimes \nu) \times L^1_{R^i}(\mu \otimes \nu)$ be an element of \mathcal{F}_A . Then by the parametric version of the Carathéodory's Theorem (cf. Castaing-Valadier [4, pp. 100–102], Maruyama [11, § 4]), there exist

$$\begin{aligned} (\alpha_i(s, t), \beta_i(s, t)) &\in \mathcal{F}_A; \quad (i=1, 2, \dots, l+2) \\ \sigma : S \times T &\rightarrow \Sigma_{l+2} \end{aligned}$$

such that

$$\begin{aligned} \alpha(s, t) &= \sum_{i=1}^{l+2} \sigma_i(s, t) \alpha_i(s, t) \\ \beta(s, t) &= \sum_{i=1}^{l+2} \sigma_i(s, t) \beta_i(s, t). \end{aligned}$$

Hence, by Corollary 1, there exist mutually disjoint measurable sets M_1, M_2, \dots, M_{l+2} such that

$$\begin{aligned} S \times T &= \bigcup_{i=1}^{l+2} M_i \\ \int_{S \times T} \alpha(s, t) d(\mu \otimes \nu) &= \int_{S \times T} \sum_{i=1}^{l+2} \sigma_i(s, t) \alpha_i(s, t) d(\mu \otimes \nu) \\ &= \int_{S \times T} \sum_{i=1}^{l+2} \alpha_i(s, t) \chi_{M_i}(s, t) d(\mu \otimes \nu) \\ \int_S \beta(s, t) d\mu &= \int_S \sum_{i=1}^{l+2} \sigma_i(s, t) \beta_i(s, t) d\mu \\ &= \int_S \sum_{i=1}^{l+2} \beta_i(s, t) \chi_{M_i}(s, t) d\mu. \end{aligned}$$

If we define

$$\begin{aligned} \tilde{\alpha}(s, t) &= \sum_{i=1}^{l+2} \alpha_i(s, t) \chi_{M_i}(s, t) \\ \tilde{\beta}(s, t) &= \sum_{i=1}^{l+2} \beta_i(s, t) \chi_{M_i}(s, t), \end{aligned}$$

then

$$(\tilde{\alpha}, \tilde{\beta}) \in \mathcal{F}_A \quad \text{and} \quad H(\alpha, \beta) = H(\tilde{\alpha}, \tilde{\beta}).$$

Thus we have completed the proof of (8).

Theorem 5. *Our variational problem (P) has a solution if A is integrably bounded. (Cf. (7).)*

References

- [1] Arkin, V. I., and V. L. Levin: Convexity of values of vector integrals, theorems on measurable choice and variational problems. *Russian Math. Surveys*, **27**, 21–85 (1972).
- [2] Castaing, C.: Sur une extension du théorème de Lyapounov. *C. R. Acad. Sc. Paris*, **260**, 3838–3841 (1965).
- [3] —: Sur une nouvelle extension du théorème de Ljapunov. *Ibid.*, **264**, 333–336 (1967).
- [4] Castaing, C., and M. Valadier: *Convex Analysis and Measurable Multifunctions*. Springer-Verlag, Berlin (1977).
- [5] Diestel, J., and J. J. Uhl, Jr.: *Vector Measures*. Amer. Math. Soc. (1977).
- [6] Dinculeanu, N.: *Vector Measures*. Pergamon Press, Oxford (1967).
- [7] Kluvánek, I., and G. Knowles: *Vector Measures and Control Systems*. North-Holland, Amsterdam (1975).
- [8] Knowles, G.: Liapunov vector measures. *SIAM J. Control*, **13**, 294–303 (1974).
- [9] Ljapunov, A.: Sur les fonctions-vecteurs complètement additives. *Izu. Akad. Nauk SSSR, Ser. Mat.*, **4**, 465–478 (1940).
- [10] Maruyama, T.: An extension of the Aumann-Perles' variational problem. *Proc. Japan. Acad.*, **55A**, 348–352 (1979).
- [11] —: Theory of measurable correspondences. *Mita J. Eco.*, **73**, 113–159 (1980) (in Japanese).
- [12] Uhl, J. J., Jr.: The range of a vector-valued measure. *Proc. Amer. Math. Soc.*, **23**, 158–163 (1969).