47. Further Results for the Solutions of Certain Third Order Differential Equations

By Tadayuki HARA
University of Osaka Prefecture

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1980)

1. **Introduction.** This paper is concerned with the differential equations of the form

(1.1) $\dddot{x}+a(t)f(x,\dot{x})\ddot{x}+b(t)g(x,\dot{x})+c(t)h(x)=p(t)+\tilde{p}(t,x,\dot{x},\ddot{x})$,

(1.2) $\dddot{x}+a(t)f(x,\dot{x})\ddot{x}+b(t)g(x,\dot{x})+c(t)h(x)=p(t)$

where $a, b, c, p, \tilde{p}, f, g, h$ are real valued functions.

The asymptotic property of solutions of third order differential equations has received a considerable amount of attention over the past two decades (cf. [1]–[8]). Many of these results are summarized in [9].

In [5], the author considered (1.1) in the case $p(t)\equiv 0$ and eatablished sufficient conditions under which all solutions of (1.1) and their first and second order derivatives are uniformly bounded and tend to zero as $t\to\infty$.

In Theorem 3.1 of this paper, sufficient conditions are given for uniform boundedness and convergence to zero of all solutions of (1.1) together with their derivatives of the first and second order. Theorem 3.1 generalizes our former result in [5]. In Theorem 3.2, necessary and sufficient conditions are given for uniform boundedness and convergence to zero of all solutions of (1.1) together with their derivatives of the first and second order.

2. **Definition and lemma.** Let us consider the following system

(2.1) $$\dot{x}=F(t,x)$$

where $F(t,x)$ is a continuous function from $[0,\infty)\times R^n$ to R^n. We denote the solution of (2.1) through (t_0, x_0) by $x(t, t_0, x_0)$.

Definition 2.1. The solutions of (2.1) are *uniformly bounded*, if for any $\alpha>0$, there exists $\beta(\alpha)>0$ such that

$$\|x(t,t_0,x_0)\|<\beta \qquad \text{for } \|x_0\|<\alpha \text{ and } t\geqq t_0\geqq 0.$$

For the proof of Theorems given below we need the following Lemma ([5, Theorem A]).

Lemma 2.1. *Suppose that there exists a Liapunov function $V(t,x)$, continuously differentiable in $[0,\infty)\times R^n$, satisfying the following conditions:*

(i) *$a(\|x\|)\leqq V(t,x)\leqq b(\|x\|)$, where $a(r), b(r)$ are continuous, increasing and positive definite functions and $a(r)\to\infty$ as $r\to\infty$.*

(ii) $\dot{V}_{(2.1)}(t, x) \equiv \lim_{h\to 0+} \frac{1}{h}\{V(t+h, x+hF(t,x)) - V(t,x)\}$

$$\leqq -\{c-\lambda_1(t)\}V(t,x)+\lambda_2(t)\{1+V(t,x)\},$$

where c is a positive constant and $\lambda_i(t)$ $(i=1,2)$ *are non-negative continuous functions satisfying*

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v} \lambda_1(s)ds < c, \tag{2.2}$$

$$\lim_{t\to\infty}\int_t^{t+1} \lambda_2(s)ds = 0. \tag{2.3}$$

Then the solutions $x(t)$ *of* (2.1) *are uniformly bounded and satisfy*

$$\lim_{t\to\infty} x(t)=0.$$

3. Assumptions and theorems. We state some assumptions on the functions appeared in (1.1)–(1.2).

Assumptions:

(I) $a(t), b(t)$ and $c(t)$ are continuously differentiable and $p(t)$ is continuous on $[0,\infty)$. $\tilde{p}(t,x,y,z)$ is continuous on $[0,\infty)\times R^3$.

(II) f, f_x, g, g_x are continuous for all $(x,y)\in R^2$ and $h(x)$ is continuously differentiable for all $x\in R^1$.

(III) $0<a_0\leqq a(t)\leqq A,\quad 0<b_0\leqq b(t)\leqq B,\quad 0<c_0\leqq c(t)\leqq C$ for $t\in[0,\infty)$.

(IV) $0<\delta\leqq h(x)/x \qquad (x\neq 0)$.

(V) $0<f_0\leqq f(x,y)\leqq \bar{f}$ for all (x,y) and $0<g_0\leqq \dfrac{g(x,y)}{y}\leqq \bar{g}$ for all $y\neq 0$ and x.

(VI) $yf_x(x,y)\leqq 0,\quad g_x(x,y)\leqq 0 \qquad$ for all $(x,y)\in R^2$.

(VII) $h'(x)\leqq h_1<\dfrac{a_0b_0f_0g_0}{C} \qquad$ for all $x\in R^1$.

(VIII) $\dfrac{\mu_2}{4c_0}\left\{A(\bar{f}-f_0)+\dfrac{B}{\mu_1}(\bar{g}-g_0)\right\}<\delta$

where μ_1 and μ_2 are arbitrarily fixed constants satisfying

$$\frac{Ch_1}{b_0g_0}<\mu_1<a_0f_0, \qquad 0<\mu_2<\frac{a_0b_0f_0g_0-Ch_1}{Af_0}.$$

(IX) $\limsup_{(t,v)\to(\infty,\infty)} \dfrac{1}{v}\displaystyle\int_t^{t+v}\{|a'(s)|+b'_+(s)+|c'(s)|\}ds<\gamma$

where γ is a small positive constant whose magnitude depends only on the constants appeared in (III)–(VIII) and $b'_+(t)=\max(b'(t),0)$.

(X) $\lim_{t\to\infty} e^{-t}\displaystyle\int_0^t e^s p(s)ds=0$.

(XI) $|\tilde{p}(t,x,y,z)|\leqq p_1(t)\{1+(x^2+y^2+z^2)^{1/2}\}+\Delta(x^2+y^2+z^2)^{1/2}$.

where Δ is a positive constant and $p_1(t)$ is a non-negative function.

(XII) $\lim_{t\to\infty}\displaystyle\int_t^{t+1} p_1(s)ds=0$.

The following results will be established:

Theorem 3.1. *Suppose that the assumptions* (I) *through* (XII) *hold. Then there exists a constant* $\varepsilon=\varepsilon(A, a_0, B, b_0, C, c_0, \delta, \bar{f}, f_0, \bar{g}, g_0, h_1)>0$ *such that if* $\Delta\leqq\varepsilon$ *then the solutions* $x(t)$ *of* (1.1) *and their derivatives* $\dot{x}(t)$ *and* $\ddot{x}(t)$ *are uniformly bounded and satisfy*

$$\lim_{t\to\infty}(x(t), \dot{x}(t), \ddot{x}(t))=(0, 0, 0).$$

Theorem 3.2. *Suppose that the assumptions* (I) *through* (IX) *hold. Then the solutions* $x(t)$ *of* (1.2) *and their derivatives* $\dot{x}(t)$ *and* $\ddot{x}(t)$ *are uniformly bounded and satisfy*

$$\lim_{t\to\infty}(x(t), \dot{x}(t), \ddot{x}(t))=(0, 0, 0)$$

if and only if (X) *holds.*

4. Proof of theorems. Proof of Theorem 3.1. The equation (1.1) is equivalent to the following system of differential equations:

$$(4.1)\quad \begin{cases} \dot{x}=y \\ \dot{y}=z+P(t) \\ \dot{z}=-a(t)f(x, y)z-b(t)g(x, y)-c(t)h(x)+P(t)\{1-a(t)f(x, y)\} \\ \qquad +\tilde{p}(t, x, y, z+P(t)) \end{cases}$$

where $P(t)\equiv e^{-t}\int_0^t e^s p(s)ds$. Note that the assumption (X) implies $P(t)\to 0$ as $t\to\infty$.

Consider a Liapunov function defined as follows

$$(4.2)\qquad V(t, x, y, z)=V_1(t, x, y, z)+V_2(t, x, y, z)+V_3(t, x, y, z)$$

where V_1, V_2, and V_3 are defined by

$$(4.3)\qquad 2V_1=2\mu_1 c(t)\int_0^x h(\xi)d\xi+2c(t)h(x)y+2b(t)\int_0^y g(x, \eta)d\eta$$
$$+2\mu_1 a(t)\int_0^x f(x, \eta)\eta d\eta+2\mu_1 yz+z^2,$$

$$(4.4)\qquad 2V_2=\mu_2 b(t)g_0 x^2+2a(t)f_0 c(t)\int_0^x h(\xi)d\xi+a^2(t)f_0^2 y^2$$
$$-\mu_2 y^2+2b(t)\int_0^x g(x, \eta)d\eta+z^2+2\mu_2 a(t)f_0 xy$$
$$+2\mu_2 xz+2a(t)f_0 yz+2c(t)h(x)y,$$

$$(4.5)\qquad 2V_3=2a^2(t)f_0\int_0^y f(x, \eta)\eta d\eta-a^2(t)f_0^2 y^2,$$

and $\mu_1>0$, $\mu_2>0$ are two arbitrarily fixed constants such that

$$\frac{Ch_1}{b_0 g_0}<\mu_1<a_0 f_0, \qquad 0<\mu_2<\frac{a_0 b_0 f_0 g_0-Ch_1}{Af_0},$$

$$\delta>\frac{\mu_2}{4c_0}\left\{A(\bar{f}-f_0)+\frac{B}{\mu_1}\mu_1(\bar{g}-g_0)\right\}.$$

In [5] we showed the following property of $V(t, x, y, z)$:

$$(4.6)\qquad k_1(x^2+y^2+z^2)\leqq V(t, x, y, z)\leqq k_2(x^2+y^2+z^2)$$

for all $t\geqq 0$ and $(x, y, z)\in R^3$, where k_1 and k_2 are certain positive constants.

Along the solution of (4.1), we have

$$(4.7)\quad \dot{V}_{(4.1)}=-W(t,x,y,z)+2b(t)y\int_0^y g_x(x,\eta)d\eta$$
$$+a(t)\{\mu_1+a(t)f_0\}y\int_0^y f_x(x,\eta)\eta d\eta$$
$$+\{\mu_1c'(t)+(a'(t)c(t)+a(t)c'(t))f_0\}\int_0^x h(\xi)d\xi$$
$$+2c'(t)h(x)y+2b'(t)\int_0^y g(x,\eta)d\eta+\mu_1a'(t)\int_0^y f(x,\eta)\eta d\eta$$
$$+\frac{1}{2}\mu_2b'(t)g_0x^2+\mu_2a'(t)f_0xy+a'(t)f_0yz$$
$$+2a(t)a'(t)f_0\int_0^y f(x,\eta)\eta d\eta$$
$$+\{\mu_2x+(\mu_1+a(t)f_0)y+2z\}\tilde{p}(t,x,y,z+P(t))$$
$$+[\mu_2\{1+a(t)(f_0-f(x,y))\}x+(a(t)f_0+\mu_1-\mu_2)y$$
$$+\{a(t)(f_0-2f(x,y))+2+\mu_1\}z+2b(t)g(x,y)$$
$$+2c(t)h(x)]P(t)$$

where

$$W=\mu_2c(t)xh(x)+\{a(t)f(x,y)-\mu_1\}z^2+a(t)\{f(x,y)-f_0\}z^2$$
$$+[\{\mu_1b(t)g_0-c(t)h'(x)\}+\{a(t)b(t)f_0g_0-c(t)h'(x)-\mu_2a(t)f_0\}]y^2$$
$$+a(t)b(t)f_0\left\{\frac{g(x,y)}{y}-g_0\right\}y^2+\mu_1b(t)\left\{\frac{g(x,y)}{y}-g_0\right\}y^2$$
$$+\mu_2b(t)\left\{\frac{g(x,y)}{y}-g_0\right\}xy+\mu_2a(t)\{f(x,y)-f_0\}xz.$$

Using the assumptions (III)–(VIII), we can find positive numbers k_3, k_4 and k_5 such that

$$\dot{V}_{(4.1)}\leqq-2k_3(x^2+y^2+z^2)+k_4\{|a'(t)|+b'_+(t)+|c'(t)|\}(x^2+y^2+z^2)$$
$$+k_5(x^2+y^2+z^2)^{1/2}\{|\tilde{p}(t,x,y,z+P(t))|+|P(t)|\}.$$

The assumption (XI) implies that

$$|\tilde{p}(t,x,y,z+P(t))|\leqq P_1(t)\{1+2|P(t)|\}+\sqrt{2}\,\Delta|P(t)|$$
$$+\sqrt{2}\,P_1(t)(x^2+y^2+z^2)^{1/2}+\sqrt{2}\,\Delta(x^2+y^2+z^2)^{1/2}.$$

Then we obtain the following estimate with some positive constants k_6 and k_7;

$$\dot{V}_{(4.1)}\leqq-2k_3(x^2+y^2+z^2)+k_4\{|a'(t)|+b'_+(t)+|c'(t)|\}(x^2+y^2+z^2)$$
$$+k_6\{p_1(t)+|P(t)|\}\{(x^2+y^2+z^2)^{1/2}+(x^2+y^2+z^2)\}$$
$$+\Delta k_7(x^2+y^2+z^2).$$

Let Δ be fixed, in what follows, to satisfy

$$(4.8)\qquad \Delta\leqq\frac{k_3}{k_7}.$$

Using the inequalities (4.6) and (4.8), we have

$$(4.9)\qquad \dot{V}_{(4.1)}\leqq-\frac{k_3}{k_2}V+\frac{k_4}{k_1}\{|a'(t)|+b'_+(t)+|c'(t)|\}V$$
$$+k_6\{p_1(t)+|P(t)|\}\left\{\left(\frac{V}{k_1}\right)^{1/2}+\left(\frac{V}{k_1}\right)\right\}.$$

Assume that

$$\limsup_{(t,v)\to(\infty,\infty)} \frac{1}{v}\int_t^{t+v} \{|a'(s)|+b'_+(s)+|c'(s)|\}ds<\frac{k_1k_3}{k_2k_4}.$$

Since $|P(t)|\to 0$ as $t\to\infty$, we have

$$\int_t^{t+1} |P(s)|\, ds\to 0$$

as $t\to\infty$. Now Lemma 2.1 is used to prove that the solutions $(x(t), y(t), z(t))$ of (4.1) are uniformly bounded and satisfy

$$\lim_{t\to\infty} (x(t), y(t), z(t))=(0, 0, 0).$$

Therefore the solution $x(t)$ of (1.1) and its derivatives $\dot{x}(t), \ddot{x}(t)$ are uniformly bounded and

$$\lim_{t\to\infty} (x(t), \dot{x}(t), \ddot{x}(t))=(0, 0, 0),$$

because $|P(t)|\to 0$ as $t\to\infty$. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. The sufficiency of (X) follows from Theorem 3.1. We prove the necessity of (X).

Let $x(t)$ be the solution of (1.2) such that

$$\lim_{t\to\infty} (x(t), \dot{x}(t), \ddot{x}(t))=(0, 0, 0).$$

Then from (1.2) we have

$$\begin{aligned}\left|e^{-t}\int_0^t e^s p(s)ds\right| &\leqq \left|e^{-t}\int_0^t e^s \dddot{x}(s)ds\right| \\ &\quad+\left|e^{-t}\int_0^t e^s a(s)f(x(s), \dot{x}(s))\ddot{x}(s)ds\right| \\ &\quad+\left|e^{-t}\int_0^t e^s b(s)g(x(s), \dot{x}(s))ds\right| \\ &\quad+\left|e^{-t}\int_0^t e^s c(s)h(x(s))ds\right| \\ &\leqq |\ddot{x}(t)|+e^{-t}|\ddot{x}(0)|+e^{-t}\int_0^t e^s|\ddot{x}(s)|\,ds \\ &\quad+A\bar{f}e^{-t}\int_0^t e^s|\ddot{x}(s)|\,ds \\ &\quad+B\bar{g}e^{-t}\int_0^t e^s|\dot{x}(s)|\,ds \\ &\quad+Ce^{-t}\int_0^t e^s|h(x(s))|\,ds.\end{aligned}$$

The assumptions (IV) and (VII) imply $h(0)=0$, thus $h(x)\to 0$ as $x\to 0$. Hence we have

$$\lim_{t\to\infty}\left|e^{-t}\int_0^t e^s p(s)ds\right|=0.$$

This completes the proof of Theorem 3.2.

References

[1] E. N. Chukwu: On the boundedness of the solutions of third order differential equations. Ann. Mat. Pura Appl., **104**, 123–149 (1975).

[2] J. O. C. Ezeilo: Stability results for the solutions of some third and fourth order differential equations. Ann. Mat. Pura Appl., **66**, 233–249 (1964).
[3] ——: A generalization of some boundedness results by Reissig and Tejumola. J. Math. Anal. Appl., **41**, 411–419 (1973).
[4] T. Hara: On the asymptotic behavior of the solutions of some third and fourth order non-autonomous differential equations. Publ. RIMS, Kyoto Univ., **9**, 649–673 (1974).
[5] ——: On the asymptotic behavior of solutions of certain non-autonomous differential equations. Osaka J. Math., **12**, 267–282 (1975).
[6] K. E. Swick: Asymptotic behavior of the solutions of certain third order differential equations. SIAM J. Appl. Math., **19**, 96–102 (1970).
[7] H. O. Tejumola: A note on the boundedness and the stability of solutions of certain third-order differential equatoins. Ann. Mat. Pura Appl., **92**, 65–75 (1972).
[8] M. Yamamoto: Further results for the solutions of certain third order non-autonomous differential equations. Proc. Japan Acad., **49**, 317–322 (1973).
[9] R. Reissig, G. Sansone, and R. Conti: Non-linear Differential Equations of Higher Order. Noordhoff, Leyden (1974).