13. Duality Theorems for Symmetric Differential Forms

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In this paper, duality theorems for symmetric (differential) forms are formulated and proved, which are generalizations of the duality of plane curves, i.e. the theorem to the effect that the dual curve of the dual curve of C coincides with C itself.

Our duality theorems include the duality for space curves given by H. Weyl and J. Weyl in [4, chap. 1].

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§ 1. Let k be a field containing $\boldsymbol{Q}$ and $\mathfrak{K}/k$ be a field extension such that k is algebraically closed in $\mathfrak{K}$. For simplicity, we assume $\mathfrak{K}$ has a transcendence basis $\xi_1, \cdots, \xi_m$ over k. Then, the ring of symmetric (differential) forms of $\mathfrak{K}$ over k SF $(\mathfrak{K}/k)$ is written in the form

$$\mathfrak{K}[d\xi_1, \cdots, d\xi_m, d^2\xi_1, \cdots, d^i\xi_j, \cdots],$$

which is isomorphic to the polynomial ring of independent variables $d\xi_1, \cdots, d\xi_m, \cdots, d^i\xi_j, \cdots$ over $\mathfrak{K}$, where d is the symmetric derivation (see [1], [2]).

Thus, SF $(\mathfrak{K}/k)$ has no zero-divisors and its field of fractions is denoted by QSF $(\mathfrak{K}/k)$. We introduce $\mathfrak{D}$: QSF $(\mathfrak{K}/k)\to$QSF $(\mathfrak{K}/k)$ by $\mathfrak{D}(\omega_1/\omega_2)=(\omega_2 d\omega_1-\omega_1 d\omega_2)/\omega_2^2$ where $\omega_1, \omega_2 \in$ SF $(\mathfrak{K}/k)$. Then $\mathfrak{D}$ is well defined and k-linear. Further, $\mathfrak{D}$ satisfies the Leibniz rule, i.e. $\mathfrak{D}(\psi\cdot\omega)=\mathfrak{D}(\psi)\cdot\omega+\psi\cdot\mathfrak{D}(\omega)$ for any $\psi, \omega \in$ QSF $(\mathfrak{K}/k)$.

For simplicity, d is again used to denote $\mathfrak{D}$: QSF $(\mathfrak{K}/k)\to$QSF $(\mathfrak{K}/k)$.

Definition. For any $\omega_1, \cdots, \omega_l \in$ QSF $(\mathfrak{K}/k)$, we define $W(\omega_1, \cdots, \omega_l)$ to be the determinant of the matrix $[d^{i-1}\omega_j]_{1\le i,j\le l}$, which is called the *Wronskian form associated with* $\omega_1, \cdots, \omega_l$.

Proposition 1. (i) *For any* $\psi \in$ QSF $(\mathfrak{K}/k)$,

$$W(\psi\omega_1, \cdots, \psi\omega_l)=\psi^l W(\omega_1, \cdots, \omega_l).$$

(ii) $W(1, \omega_2, \cdots, \omega_l)=W(d\omega_2, \cdots, d\omega_l)$.

(iii) *If* $\omega_1, \cdots, \omega_l$ *are k-linearly dependent, then* $W(\omega_1, \cdots, \omega_l)=0$.

Proofs of the above results are easy and omitted.

By using (i) and (ii), we can compute $W(\omega_1, \cdots, \omega_l)$ as follows:

$$W(\omega_1, \cdots, \omega_l)=\omega_1^l W\left(d\left(\frac{\omega_2}{\omega_1}\right), \cdots, d\left(\frac{\omega_l}{\omega_1}\right)\right)$$

$$=\omega_1^l\left(d\frac{\omega_2}{\omega_1}\right)^{l-1}\cdot \quad W\left(d\frac{d\left(\frac{\omega_3}{\omega_1}\right)}{d\left(\frac{\omega_2}{\omega_1}\right)},\cdots,d\frac{d\left(\frac{\omega_l}{\omega_1}\right)}{d\left(\frac{\omega_2}{\omega_1}\right)}\right)=\cdots.$$

Theorem 1. *If $\omega_1,\cdots,\omega_l$ are k-linearly independent, then* $W(\omega_1,\cdots,\omega_l)\neq 0$.

Proof. We first consider the case $l=2$. Then we may assume $\omega_1,\omega_2\in \mathrm{SF}(\mathfrak{K}/k)$, which are k-linearly independent. In this case, we shall prove that $W(\omega_1,\omega_2)\neq 0$.

By N_j denoting Exponents $(a_{lj},\cdots,a_{rj},\cdots)\in\bigoplus^{\infty}\boldsymbol{Z}_0$, $\boldsymbol{Z}_0=\{\alpha\in\boldsymbol{Z}\mid\alpha\geq 0\}$ we introduce the following notation (see [2]):

(i) $(d\xi_j)^{N_j}=(d\xi_j)^{a_{1j}}(d^2\xi_j)^{a_{2j}}\cdots(d^r\xi_j)^{a_{rj}}\cdots$.

(ii) Letting $L=(N_1,N_2,\cdots,N_m)$, we put

$$(d\xi)^L=(d\xi_1)^{N_1}\cdots(d\xi_m)^{N_m}.$$

(iii) $(N_1,\cdots,N_m)<(N_1',\cdots,N_m')$ is defined by the existence of i such that $N_i<N_i'$, $N_{i+1}=N_{i+1}',\cdots,N_m=N_m'$, in which $N_i=(a_1,a_2,\cdots,a_r,\cdots)<N_i'=(b_1,b_2,\cdots,b_r,\cdots)$ is defined by the existence of j such that $a_j<b_j$, $a_{j+1}=b_{j+1},\cdots,a_r=b_r$, (for any $r>j$).

(iv) For $\omega=\sum\varphi_L(d\xi)^L\in\mathrm{SF}(\mathfrak{K}/k)\backslash\{0\}$, define $H(\omega)$ to be $\max\{L\mid\varphi_L\neq 0\}$. $\omega^*=\varphi_{H(\omega)}(d\xi)^{H(\omega)}$ is said to be *the highest part* of ω.

(v) If $L=(N_1,N_2,\cdots,N_m)$, we put $s(L)=\max\{j\mid N_j\neq 0\}$, 0 denoting $(0,0,\cdots)$. When $L=(0,0,\cdots,0)$, $s(L)$ is defined to be 0.

(vi) If $N=(a_1,a_2,\cdots)$, we put $r(N)=\max\{j\mid a_j\neq 0\}$. When $N=(0,0,\cdots)$, we define $r(N)$ to be 0.

(vii) If $r(N)>0$, then dN is defined to be $(a_1,\cdots,a_{r-1},a_r-1,1,0,0,\cdots)$ where $r=r(N)$ and $N=(a_1,a_2,\cdots,a_r,0,\cdots)$. Thus $H((d\xi_1)^{N_1})=N_1$ and $H(d(d\xi_1)^{N_1})=dN_1$, if $N_1\neq 0$.

(viii) If $L=(N_1,N_2,\cdots,N_m)\neq 0$, then define dL to be $(N_1,N_2,\cdots,N_{s-1},dN_s,0,\cdots,0)$ where $s=s(L)$. Thus $H(d(d\xi)^L)=dL$, if $L\neq 0$.

Lemma 1. (I) *If $\omega_1,\omega_2\neq 0$ such that $H(\omega_1)>H(\omega_2)$, then $W(\omega_1,\omega_2)\neq 0$ and $H(W(\omega_1,\omega_2))=dH(\omega_1)+H(\omega_2)$.*

(II) *If $H(\omega_1)=H(\omega_2)=H$, and $\omega_1^*=(d\xi)^H$, $\omega_2^*=\varphi(d\xi)^H$ with $\varphi\notin k$, then $W(\omega_1,\omega_2)\neq 0$ and $H(W(\omega_1,\omega_2))=2H+H((d\varphi))$.*

Proof is easy.

If $W(\omega_1,\omega_2)=0$, then by the above lemma, $H(\omega_1)=H(\omega_2)=H$ and $\omega_1^*/\omega_2^*\in k$. Thus, there exists $\alpha\in k$ such that $H(\omega_1-\alpha\omega_2)<H$. Hence $W(\omega_1-\alpha\omega_2,\omega_2)\neq 0$ if $\omega_1-\alpha\omega_2\neq 0$. But

$$W(\omega_1-\alpha\omega_2,\omega_2)=W(\omega_1,\omega_2)-\alpha W(\omega_2,\omega_2)=0.$$

This is a contradiction. Therefore, $W(\omega_1,\omega_2)=0$ with $\omega_2\neq 0$ implies that $\omega_1=\alpha\omega_2$ for some $\alpha\in k$.

Now, we prove Theorem 1 by induction on l. If $\omega_1,\omega_2,\cdots,\omega_l$ are

k-linearly independent, then so are $\omega_2/\omega_1, \cdots, \omega_l/\omega_1$. This implies that $d(\omega_2/\omega_1), \cdots, d(\omega_l/\omega_1)$ are k-linearly independent. As a matter of fact, if there exist $\alpha_1, \cdots, \alpha_{l-1} \in k$ such that $\sum \alpha_j d(\omega_{j+1}/\omega_1) = 0$, then $d(\sum \alpha_j \omega_{j+1}/\omega_1) = 0$. From what we have proved above, it follows that $\sum \alpha_j \omega_{j+1} = \alpha_0 \omega_1$ for some α_0, i.e. $\omega_1, \cdots, \omega_l$ are k-linearly dependent, a contradiction. Therefore, by induction hypothesis, $W(d(\omega_2/\omega_1), \cdots, d(\omega_l/\omega_1)) \neq 0$. Recalling that $W(\omega_1, \omega_2, \cdots, \omega_l) = \omega_1^l W(d(\omega_2/\omega_1), \cdots, d(\omega_l/\omega_1))$, we complete the proof of Theorem 1.

§2. We fix $1, x_1, \cdots, x_n \in \mathfrak{K}$ which are k-linearly independent. Putting $W(x) = W(x_1, x_2, \cdots, x_n)$ and $W((dx)^i) = W(dx_1, \cdots, dx_{i-1}, dx_{i+1}, \cdots, dx_n)$, we define u_i to be $(-1)^i W((dx)^i)/W(x)$ for $1 \leq i \leq n$. Then since

$$\begin{vmatrix} d^p x_1 & d^p x_2 & d^p x_n \\ dx_1 & dx_2 & dx_n \\ & \cdots & \\ d^{n-1}x_1 & d^{n-1}x_2 & d^{n-1}x_n \end{vmatrix} = \begin{cases} W(x) & \text{if } p=0, \\ 0 & \text{if } 1 \leq p \leq n-1, \end{cases}$$

we obtain $\sum x_i u_i + 1 = 0$ and $\sum d^p x_i \cdot u_i = 0$ for $1 \leq p \leq n-1$. Setting $d^p x \mid d^q u = \sum_{i=1}^{n} d^p x_i \cdot d^q u_i$, we obtain the next lemma.

Lemma 2. $(\mathrm{I})_{p,q}$ *$d^p x \mid d^q u = 0$ if $1 \leq p+q \leq n-1$.*

$(\mathrm{II})_p$ *$d^p x \mid d^{n-p} u = (-1)^p \omega_x$, where $\omega_x = W(dx)/W(x)$, and $W(dx) = W(dx_1, \cdots, dx_n)$.*

Proof. We first prove $\mathrm{I}_{p,q}$ by induction on q. If $q=0$, this was already proved. Assume $\mathrm{I}_{p,r}$ for $r \leq q-1$. Then for $p+q \leq n-1$, $d^p x \mid d^{q-1} u = -1$ or 0, hence $0 = d(d^p x \mid d^{q-1} u) = d^{p+1} x \mid d^{q-1} u + d^p x \mid d^q u$. Thanks to $d^{p+1} x \mid d^{q-1} u = 0$ by $\mathrm{I}_{p+1,q-1}$, we obtain $d^p x \mid d^q u = 0$.

Next, we prove II_p by induction on $n-p$. By the expansion of $W(dx)$, we have

$$\frac{W(dx)}{W(x)} = (-1)^n d^n x \mid u,$$

hence II_n holds. If II_p is true, then from $\mathrm{I}_{p-1,n-p}$, it follows that $0 = d(d^{p-1} x \mid d^{n-p} u) = d^p x \mid d^{n-p} u + d^{p-1} x \mid d^{n-p+1} u$. Hence, $d^{p-1} x \mid d^{n-p+1} u = -d^p x \mid d^{n-p} u = (-1)^{p-1} W(dx)/W(x)$. Q.E.D.

Define the next matrices:

$$X(1, x) = \begin{pmatrix} 1, & x_1, \cdots, & x_n \\ 0, & dx_1, \cdots, & dx_n \\ 0, & d^{n-1}x_1, \cdots, & d^n x_n \end{pmatrix},$$

$$X(x) = \begin{pmatrix} x_1, \cdots, & x_n \\ \cdots & \\ d^{n-1}x_1, \cdots, & d^{n-1}x_n \end{pmatrix}, \qquad X(dx) = \begin{pmatrix} dx_1, \cdots, & dx_n \\ \cdots & \\ d^n x_1, \cdots, & d^n x_n \end{pmatrix}.$$

Then by Lemma 2,

$$X(1,x)\cdot {}^tX(1,u)=\begin{pmatrix} 0 & & x|d^nu \\ & dx|d^{n-1}u & \\ & \cdot^{\cdot^{\cdot}} & * \\ d^nx|u & & \end{pmatrix}=\begin{pmatrix} 0 & & \omega_x \\ & \cdot^{\cdot^{\cdot}} & -\omega_x \\ & \cdot^{\cdot^{\cdot}} & * \\ (-1)^n\omega_x & & \end{pmatrix}$$

and

$$X(x)\cdot {}^tX(du)=\begin{pmatrix} 0 & & x|d^nu \\ & dx|d^{n-1}u & \\ & \cdot^{\cdot^{\cdot}} & * \\ d^{n-1}x|du & & \end{pmatrix}=\begin{pmatrix} 0 & & \omega_x \\ & \cdot^{\cdot^{\cdot}} & -\omega_x \\ & \cdot^{\cdot^{\cdot}} & * \\ (-1)^{n-1}\omega_x & & \end{pmatrix}.$$

Hence, $W(dx)W(du)=\omega_x^{n+1}$ and $W(x)W(du)=\omega_x^n$. Similarly, $W(u)W(dx)=(-1)^n\omega_x^n$.

Proposition 2. *$1, u_1, \cdots, u_n$ are k-linearly independent.*

Proof. Since $W(dx)\neq 0$, $W(x)\neq 0$, it follows that $W(du)=\omega_x^{n+1}/W(dx)=W(dx)^n/W(x)^{n+1}\neq 0$. Q.E.D.

Theorem 2 (First duality theorem).

$$x_j=(-1)^jW((du)^j)/W(u) \qquad \textit{for all } 1\leq j\leq n.$$

Proof. By $\mathrm{I}_{0,q}$ of Lemma 2 and $x|u+1=0$, we have

$$x|u=\sum u_j\cdot x_j=-1,$$
$$dx|u=\sum du_j\cdot x_j=0,$$
$$\vdots$$
$$d^{n-1}x|u=\sum d^{n-1}u_j\cdot x_j=0.$$

Thus, $x_j=(-1)^jW((du)^j)/W(u)$ for all j.

Proposition 3. $\omega_u=\dfrac{W(du)}{W(u)}=(-1)^n\omega_x.$

Proof. By $X(dx)\cdot {}^tX(u)$, we have

$$W(dx)W(u)=(-1)^n\omega_x^n.$$

Thus

$$\omega_u=\frac{W(du)}{W(u)}=\frac{W(dx)W(du)}{W(dx)W(u)}=(-1)^n\omega_x. \qquad \text{Q.E.D.}$$

§ 3. Let V be a k-vector space with basis $\{e_0,\cdots,e_n\}$ and $g,f: V\to V$ be linear maps such that

$$(f\circ {}^tg)(e_i)=(-1)^{n-i}\omega e_{n-i}+\alpha_{n-i+1,i}e_{n-i+1}+\cdots+\alpha_{n,i}e_n,$$

where ${}^tg$ denotes the dual map of g, $\alpha_{j,i}\in k$, $\omega\in k$, and $0\leq i\leq n$.

Now, we use the following notation to denote vectors in the exterior algebra $\wedge^{\cdot}V$ of V over k: for any subset I of $N=\{0,1,\cdots,n\}$, we assume that the elements are arranged by the order of the natural numbers, i.e. if I is $\{i_1,\cdots,i_s\}$, then $i_1<\cdots<i_s$.

cI is defined to be the complement of I in N. For $I=\{i_1,\cdots,i_s\}$, we put $e_I=e_{i_1}\wedge\cdots\wedge e_{i_s}$ and define sgn (I) by $e_I\wedge e_{cI}=\operatorname{sgn}(I)e_N$.

Corresponding to $f: V\to V$, we have $f_s: \wedge^sV\to\wedge^sV$ defined by $f_s(e_{i_1}\wedge\cdots\wedge e_{i_s})=f(e_{i_1})\wedge\cdots\wedge f(e_{i_s})$, for any $I=\{i_1,\cdots,i_s\}\subseteq N$.

Writing $f(e_i)=\sum a_{ji}e_j$ and $f_s(e_I)=\sum a_{JI}^{(s)}e_J$, we see that $a_{JI}^{(s)}$

$=\det [a_{j_p,i_q}]_{1\le p,q\le s}$ where $I=\{i_1,\cdots,i_s\}$ and $J=\{j_1,\cdots,j_s\}$. Letting $a^{*(s)}_{I,J}=\operatorname{sgn}(I)\operatorname{sgn}(J)a^{(n+1-s)}_{cJ,cI}$, we define $f^*_s:\wedge^s V\to\wedge^s V$ by $f^*_s(e_I)=\sum a^{*(s)}_{J,I}e_J$. Then $f^*_s\circ f_s=(\det f)\cdot id$, $\det f$ denoting the determinant of the matrix corresponding to f.

From the hypothesis, we have

$$(f\circ {}^tg)_s(e_{\{0,1,\dots,s-1\}})=(-1)^{ns}\omega^s e_{c\{0,1,\dots,n-s\}}.$$

Hence, $\det f\cdot {}^tg_s(e_{\{0,1,\dots,s-1\}})=(-1)^{ns}\omega^s f^*_s e_{c\{0,1,\dots,n-1\}}$ and so $\det f\cdot\sum b^{(s)}_{P,I}e_I=(-1)^{ns}\omega^s\sum a^{*(s)}_{I,cQ}e_I$, where $P=\{0,1,\cdots,s-1\}$, $Q=\{0,1,\cdots,n-s\}$, and $I=\{i_1,\cdots,i_s\}$.

Thus $\det f\cdot b^{(s)}_{P,I}=(-1)^{ns}\omega^s\operatorname{sgn}(I)a^{(n+1-s)}_{Q,cI}$.

Applying the above formula to $X(1,x)\cdot{}^tX(1,u)$, we obtain the next theorem.

Theorem 3 (Second duality theorem). *For any $I=\{i_1,\cdots,i_s\}$ and $cI=\{j_1,\cdots,j_{n-s+1}\}$, put $M_I(x)=W(x_{i_1},\cdots,x_{i_s})$ and $M_{cI}(u)=W(u_{j_1},\cdots,u_{j_{n-s+1}})$, in which x_0 and u_0 denote 1. Then*

$$W(dx)M_I(u)=(-1)^{ns}\omega^s_x\operatorname{sgn}(I)M_{cI}(x).$$

§ 4. Let Z be a non-singular variety such that the field of rational functions is $\mathfrak{K}$. Suppose that $x_0=1,x_1,\cdots,x_n$ are regular at p, i.e. $x_0,x_1,\cdots,x_n\in\mathcal{O}_{Z,p}$. We have the $E\left(=\sum_{j=0}^{n}kx_j\right)$-gap sequence at p. In other words, there exists a sequence of (generic) quadric transformations $f_j:Z_j\to Z_{j-1}$ whose center are points $p_j\in Z_j$ which are general points of $f_j^{-1}(p_{j-1})$ such that $p_0=p$, $Z_0=Z$ and $1\le j\le n$. Letting $q=p_n$ and $\mu=f_1\circ\cdots\circ f_n:Z_n\to Z_0$, we have $\mu^*:\mathcal{O}_{Z,p}\to\mathcal{O}_{Z_n,q}$ which satisfies that μ^*E has a basis $\{y_0=1,y_1,\cdots,y_n\}$ such that $\nu_q(y_0)=0=a_1<\nu_q(y_1)=a_2<\cdots<\nu_q(y_n)=a_{n+1}$, where $\nu_q(y)$ is the order of y at q (see [1], [2]). By definition of Theorem 2 in [2], $\{1,a_2,\cdots,a_n\}$ becomes the E-gap sequence at p. We define the dual space of E by $\omega^{(n)}(E)=$the space spanned by $W(z_1,\cdots,z_n)$ where $z_j\in E$. Then $\mu^*(\omega^{(n)}(E))=\omega^{(n)}(\mu^*E)$ has the basis $\{\omega_i=W(y_0,\cdots,y_{i-1},y_{i+1},\cdots,y_n)\}_i$, in which

$$\nu_q(\omega_i)=\sum_{j\ne i+1}a_j-n(n-1)/2\qquad\text{for all } 0\le i\le n.$$

Then letting $\beta_i=\nu_q(\omega_{n-i})$ for $0\le i\le n$, we have the sequence $(\beta_0,\beta_1,\cdots,\beta_n)$, which is considered as the $\omega^{(n)}(E)$-gap sequence at p. The sequence $B=(0,b_2,\cdots,b_{n+1})$ defined by $b_i=\beta_{i-1}-\beta_0$ for $2\le i\le n+1$ is the reduced sequence of $(\beta_0,\cdots,\beta_n)$ and it is said to be the *dual sequence* of $A=(0,a_2,\cdots,a_n)$. B is denoted by A^*. Then $b_j=a_{n+1}-a_{n+1-j}$, for all $1\le j\le n+1$, and $A^{**}=A$.

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