94. The Kodaira Dimension of Certain Fiber Spaces

By Yujiro KAWAMATA
Department of Mathematics, University of Tokyo
(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1979)

In this paper we shall sketch a proof of the following theorem. Details will appear elsewhere.

Theorem. *Let $f: X \to Y$ be a morphism of non-singular projective algebraic varieties defined over the complex number field C with a general fiber F. We assume that F is irreducible and satisfies one of the following three conditions*:

(1) $\dim F = 1$,

(2) $\dim F = 2$ *and* $\kappa(F) \neq 2$,

(3) *F is an abelian variety.*

Then $\kappa(X) \geqslant \kappa(Y) + \kappa(F)$. Moreover, if $\kappa(F) = 0$, then $\kappa(X/Y) \geqslant \mathrm{Var}(f)$, where κ and Var *denote the Kodaira dimension and the variation, respectively* (cf. [3] and [7]).

In the cases (1) and (3) the above theorem was proved in [6] and [5], respectively. But our proof is different and does not use "good" compactifications of moduli spaces.

This work was mainly prepared when the author was in Mannheim University. He wishes to express his thanks to the members there, especially to Prof. H. Popp and to Dr. E. Viehweg for valuable discussions.

Step 1. We assume that F is a $K3$ surface.

The period domain D of F is a bounded domain and its arithmetic quotient D/Γ has a Baily-Borel compactification $\overline{D/\Gamma}$ (cf. [1]). Of course the latter has nothing to do with the singular fibers of a family of F's. Let L be a free Z-module of rank 21 with a base $\{e_1, \cdots, e_{21}\}$ and with a non-degenerate inner product defined by $e_1e_{21} = e_2e_{20} = 1$, $e_3^2 = e_4^2 = \cdots = e_{19}^2 = -1$ and $e_ie_j = 0$, otherwise. Let Y_0 be a Zariski open subset of Y such that f is smooth on $X_0 = f^{-1}(Y_0)$. Let $F = F_y = f^{-1}(y)$ for a $y \in Y_0$. The polarization of X defines a homology class h on F_y and $H_2(F_y, C)/\langle h \rangle$ has a standard lattice isomorphic to L. A point p of D defines a 1-dimensional subspace in $\mathrm{Hom}(L, C)$. Let ω_p be its element such that $\omega_p(e_{21}) = 1$. If F and some Z-base of $H_2(F, Z)$ correspond to p, then ω_p defines a holomorphic 2-form on F, which we denote again by ω_p. If $q = \gamma p$ for some $\gamma \in \Gamma$, then $\omega_q = c\omega_p$, where $c = c(p)$ is an automorphic factor such that c^{19} is equal to the functional determinant. Thus each automorphic form $a(p)$ of weight k defines a section $a(p)\omega_p^k$ of $K_{X_0/Y_0}^{\otimes k}$. We have only to prove that it can be extended to a section of $K_{X/Y}^{\otimes k}$. Using the reduction step as in [6], we may assume that the group Γ is small enough and $c(p)$ defines an invertible sheaf H on $\overline{D/\Gamma}$. Then a section $a(p)\omega_p^k$ of $H^{\otimes k}$ is locally a composition of sections of H and we can prove that they define sections of $K_{X/Y}$ locally by the following lemmas.

Lemma 1. *Let $f: X \to Y$ be a morphism of non-singular projective algebraic varieties, let Y_0 be a Zariski open subset of Y, let $X_0 = f^{-1}(Y_0)$, and assume that $f|_{X_0}$ is smooth. Let F be an invertible sheaf on X and let ω be a holomorphic section of F on X_0. We assume that for any non-singular curve C, any morphism $\varphi: C \to Y$ such that $\varphi(C) \cap Y_0 \neq \phi$ and any non-singular model X_C of the closure of $X_0 \times_{Y_0} \varphi^{-1}(\varphi(C) \cap Y_0)$ in $X \times_Y C$, the pull back of ω can be extended to a section ω_C of $F \times_X X_C$ over X_C. Then, ω can be extended to a section of F over X.*

Lemma 2 (cf. [2]). *Let X be a non-singular algebraic variety and let X_0 be a Zariski open subset of X. Let $\omega \in H^0(X_0, K_{X_0})$. If $\int_{X_0} \omega \wedge \bar{\omega} < \infty$, then ω can be extended to a section of K_X.*

Step 2. When F is an abelian variety, the proof is similar. In this case D is the Siegel upper half plane.

Step 3. F is assumed to be either an Enriques surface or a hyperelliptic surface.

An automorphic form of weight k defines a section of $K_{X_0/Y_0}^{\otimes mk}$ for some positive integer m. They are also extendable.

Step 4. F is a surface with $\kappa(F) = 1$.

The Iitaka fibering offers a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & \\ {\scriptstyle f}\downarrow & & Z \\ Y & \xrightarrow{h} & \end{array}$$

where g is an elliptic fiber space and h is a family of curves. The proof is an application of Kodaira's theory of elliptic fiber spaces as in [4].

Step 5. F is a curve.

The theorem follows from the case (3), Lemma 1 and the following

Lemma 3. *Let $f: X \to Y$ be a proper morphism of non-singular algebraic varieties such that* $\dim X = 2$, $\dim Y = 1$ *and a general fiber F is an irreducible curve of genus $g \geqslant 2$. Then there is a canonically defined "Weierstrass section" of $K_{X/Y}^{\otimes g(g+1)/2} \otimes f^*(\bigwedge^g f_*(K_{X/Y}))^{-1}$.*

References

[1] W. Baily: Introductory Lectures on Automorphic Forms. Iwanami and Princeton U.P. (1973).
[2] F. Sakai: Kodaira dimensions of complements of divisors. Complex Analysis and Algebraic Geometry. Iwanami and Cambridge U.P., pp. 239–257 (1977).
[3] K. Ueno: Classification Theory of Algebraic Varieties and Compact Complex Spaces. Lect. Notes in Math., vol. 439, Springer (1975).
[4] ——: Classification of algebraic varieties. I. Compositio Math., **27**, 277–342 (1973).
[5] ——: On algebraic fiber spaces of abelian varieties. Math. Ann., **237**, 1–22 (1978).
[6] E. Viehweg: Canonical divisors and the additivity of the Kodaira dimensions for morphisms of relative dimension one. Compositio Math., **35**, 197–223 (1977).
[7] ——: Klassifikationstheorie algebraischer Varietaeten der Dimension drei (to appear in Compositio Math.).