93. The Structure of the Albanese Map of an Algebraic Variety of Kodaira Dimension Zero

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(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1979)

In this paper we shall sketch an outline of a proof of the following

Main Theorem. *Let X be a non-singular projective algebraic variety defined over the complex number field C and let $\alpha: X \to A(X)$ be the Albanese map. We assume that $\kappa(X)=0$. Then α is a fiber space.*

A *fiber space* is a morphism of non-singular projective algebaric varieties which is surjective and has connected fibers.

Corollary. *If $\kappa(X)=0$, then we have $q(X) \underset{\mathrm{def}}{=} \dim H^0(X, \Omega_X^1) \leqq \dim X$. Moreover, if the equality holds, then α is birational. Thus $\kappa(X)=0$ and $q(X)=\dim X$ give a characterization for an abelian variety up to birational equivalences.*

This work was prepared mainly when the author was in Mannheim University. He wishes to express his thanks to the members there, especially to Prof. H. Popp and to Dr. E. Viehweg for valuable discussions. He also thanks to Dr. T. Fujita who pointed out a mistake in the first version of this paper, and to Profs. S. Iitaka and K. Ueno who originated the classification theory of algebraic varieties.

The proof of Main Theorem is carried out along the standard program of classification theory of algebraic varieties (cf. [6]). The main point is the following "addition theorem".

Theorem 1. *Let $f: X \to Y$ be a fiber space and assume that $\kappa(X) \geqslant 0$ and $\kappa(Y)=\dim Y$. Then $\kappa(X)=\kappa(Y)+\kappa(F)$, where F is a general fiber.*

Beside this we have to know something about abelian varieties:

Theorem 2. *Let $f: X \to A$ be a generically finite morphism from a non-singular projective algebraic variety to an abelian variety. Then $\kappa(X) \geqslant 0$ and when we replace a birational model of X, if necessary, the Iitaka fibering $\Phi: X \to \bar{X}$ satisfies the following conditions:*

(1) *There is an abelian subvariety B of A and a general fiber of Φ is birationally equivalent to an etale cover of B.*

(2) *$\bar{X}$ is generically finite over A/B.*

(3) *$\kappa(\bar{X})=\dim \bar{X}=\kappa(X)$.*

The proof of Theorem 2 follows from Theorem "B_n" of [3]. Main

Theorem follows immediately from Theorems 1 and 2.

We shall sketch the proof of Theorem 1. By Fujita's reduction (cf. [7]) we may assume that $p_g(X)\neq 0$, hence $p_g(F)\neq 0$. By Hironaka's resolution we get the following situation: There is a divisor D of normal crossing on Y and when we put $Y_0=Y-D$, $X_0=f^{-1}(Y_0)$ and $f_0=f|_{X_0}$, then f_0 is smooth. Put $n=\dim F$. We know that the local monodromies of $R^n f_{0*}C_{X_0}$ around D are quasi-unipotent.

Proposition 1. *There is a finite Galois cover $\pi: \tilde{Y}\to Y$ with the following properties:*

(1) *$\tilde{Y}$ is non-singular.*

Let $\tilde{X}$ be a non-singular model of the closure of $\tilde{X}_0=X_0\times_Y \tilde{Y}$ in $X\times_Y \tilde{Y}$. Let $\tilde{f}: \tilde{X}\to\tilde{Y}$ be the induced morphism and put $\tilde{f}_0=\tilde{f}|_{\tilde{X}_0}$.

(2) *$\tilde{D}=\pi^{-1}(D)$ is a divisor of normal crossing.*

(3) *The local monodromies of $R^n\tilde{f}_{0*}C_{\tilde{X}_0}$ around D are unipotent.*

Theorem 1 follows from following Main Lemma and Propoition 2. In general, let X be a non-singular projective algebraic variety and let F be a locally free sheaf on X. F is said to be *semi-positive* if for any non-singular projective curve C, for any morphism $\varphi: C\to X$ and for any quotient invertible sheaf Q of φ^*F, we have $\deg_C Q\geqslant 0$.

Main Lemma. *$\tilde{f}_*K_{\tilde{X}/\tilde{Y}}$ is a locally free sheaf and semi-positive.*

Proposition 2 (cf. [4]). *Let Y be a non-singular projective algebraic variety such that $\kappa(Y)=\dim Y$ and let F be an invertible sheaf on Y. Then there exists a positive integer m such that $H^0(Y, K_Y^{\otimes m}\otimes F^{\otimes -1})\neq 0$.*

The proof of Main Lemma is an application of the theory of Hodge structures and their variations in [2] and [5]. $R^n\tilde{f}_{0*}C_{\tilde{X}_0}$ has a natural extension V which is locally free over $\tilde{Y}$. The "limit filtration F_∞" defines a locally free subsheaf F of V, which we can prove is equal to $\tilde{f}_*K_{\tilde{X}/\tilde{Y}}$. The semi-positivity of F is proved as follows: First, assume that $\varphi(C)\cap\tilde{Y}_0\neq\emptyset$. Then, by the positivity of the curvature proved by Griffiths [2], $\deg_C Q\geqslant 0$ (cf. [1]). Next, we assume that there is an irreducible component D^1 of $\tilde{D}$ such that, when we write $D_0^1=D^1\cap\operatorname{Reg}\tilde{D}$, then $\varphi(C)\subset D^1$ and $\varphi(C)\cap D_0^1\neq\emptyset$. The local monodromy around D^1 defines the "weight filtration W" on V near D^1 and the limit filtration F_∞ defines a variation of Hodge structures on $\mathrm{Gr}_W(V|_{D^1})$, which has also a Gauss-Manin connection. Then $\deg_C Q\geqslant 0$ follows just as in the former case. When $\varphi(C)$ falls into a smaller subset of $\tilde{Y}$, we have to divide V again with other weight filtrations.

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