90. A Class of General Boundary Conditions for Multi-Dimensional Diffusion Equation

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1. Let D be the upper half space $R^N_+=\{(x_1,\cdots,x_N)\in R^N|x_N>0\}$ of R^N, or a bounded open domain with smooth boundary in R^N, and let

$$\frac{\partial u}{\partial t}=Au=\sum_{1\leq i,j\leq N}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}(t,x)+\sum_{1\leq i\leq N}b_i(x)\frac{\partial u}{\partial x_i}(t,x)+c(x)u(t,x)\tag{1}$$

be a diffusion equation on D with real smooth coefficients defined on $\bar{D}=D\cup\partial D$.

Here, we should like to introduce an existence theorem for (1) with boundary conditions of type

$$Lu(x)=\tilde{A}u(x)+\delta(x)Au(x)+\partial u/\partial n(x)+\nu[u](x)=0,\tag{2}$$

where $\tilde{A}$ is an elliptic differential operator with real coefficients on ∂D

$$\tilde{A}u(x)=\sum_{0\leq|\alpha|\leq 2n}\tilde{a}_\alpha(x)\tilde{D}^\alpha u(x),\qquad x\in\partial D,\tag{3}$$

$\tilde{D}^\alpha u(x)=\partial^{|\alpha|}u(x)/\partial\xi^{\alpha_1}_{1,x}\cdots\partial\xi^{\alpha_{N-1}}_{N-1,x}$, $\alpha=(\alpha_1,\cdots,\alpha_{N-1})$. $\{\xi_{i,x}(y),\ 1\leq i\leq N\}$ is a local coordinate near $x\in\partial D$, and is also a set of bounded functions on a neighbourhood of $\bar{D}$. $\delta(x)$ is a non-positive function on ∂D. $u\to\nu[u]$ is an integro-differential operator of type

$$\nu[u](x)=(-1)^{[m/2]}\int_{\bar{D}\backslash\{x\}}\Big(u(y)-\sum_{0\leq|\alpha|\leq m}\frac{1}{\alpha!}\tilde{D}^\alpha u(x)\xi^\alpha_x(y)\Big)\nu(x,dy),\tag{4}$$

where $\xi^\alpha_x(y)=\xi^{\alpha_1}_{1,x}(y)\cdots\xi^{\alpha_{N-1}}_{N-1,x}(y)$ and $\alpha!=\alpha_1!\cdots\alpha_{N-1}!$. $\nu(x,\cdot)$ is a measure on $\bar{D}\backslash\{x\}$ such that, for each neighbourhood U_x of $x\in\partial D$,

$$\int_{U_x\backslash\{x\}}\Big(\sum_{1\leq i\leq N-1}|\xi_{i,x}(y)|^{m+1}+\xi_{N,x}(y)\Big)\nu(x,dy)+\nu(x,\bar{D}\backslash U_x)<\infty.\tag{5}$$

$\dfrac{\partial}{\partial n}$ is the inward directed normal derivative defined relative to $\{a_{ij}(x)\}$. The detailed proof of our existence theorem will be published elsewhere.

In case $m=n=1$, (2) was obtained by Wentzell [1] as a necessary condition for positive solutions of (1) on a certain set up. The sufficiency was proved by [1], Ueno [2] or Sato-Ueno [3], Bony *et al.* [4], Taira [5], Ueno [6] or [7], and others under auxiliary conditions.

The results for general m and n in this paper were motivated by the method in [7], where (*conditional*) *positive definiteness* is essential instead of the *positivity* in the case of $m=n=1$. Another motivation is a theorem of Gelfand-Vilenkin [8], where a conditionally positive generalized function F of order n, on a certain space of functions, is characterized by a form

$$(F,\varphi)=\sum_{0\leq|\alpha|\leq 2n} a_\alpha D^\alpha\varphi(0)+\int_{R^N\setminus\{0\}}\Big(\varphi(x)-\alpha(x)\sum_{0\leq|\alpha|\leq 2n-1} 1/\alpha!\, D^\alpha\varphi(0)x^\alpha\Big)\nu(dx),$$

where a_α, $\alpha(x)$ and $\nu(\cdot)$ satisfy certain conditions.

In case $m\geq 2$ or $n\geq 2$, the solutions of (1)–(2) for positive initial datas are not necessarily positive, and hence the solutions have no probabilistic meanings in the ordinary sense. But, in view of the works by Krylov [9], Miyamoto [10] and Hochberg [11], it seems that the diffusion equation with the boundary condition of type (2) will be interpreted intuitively in some natural way.

As in the case of $m\leq 1$ and $n=1$, the semigroups on the boundary also exist, and the wave equation with similar boundary conditions can be solved. These will be discussed in other articles.

2. We assume, for simplicity, that ∂D is of class C^∞ when D is bounded, and that the coefficients of A and $\tilde{A}$ and $\delta(x)$ are C^∞ functions on $\bar{D}$ and ∂D, and all derivatives of the coefficients are bounded. Moreover, we assume that A is strongly elliptic uniformly on $\bar{D}$.

In case $D=R^N_+$, we choose a C^∞ function $a(x)$ on $[0,\infty)$ such that $a(x)=1$ when $x\leq 1$, and $0<a(x)\leq a/|x|^{m+1}$ when $x>1$, and put

$$\xi_{i,x}(y)=a(|y-x|)(y_i-x_i),\qquad 1\leq i\leq N.$$

In case D is bounded, let $\{\xi_{i,x}(y)\}$ be of class C^∞, $\xi_{N,x}(y)\geq 0$ on $\bar{D}$, and $\xi_{N,x}(y)=0$ characterizes ∂D, and $\xi_{i,x}(x)=0$ for $1\leq i\leq N$.

Let $\nu_D(x,\cdot)$ and $\nu_\partial(x,\cdot)$ be the restrictions of $\nu(x,\cdot)$ to D and ∂D, and let $\nu_D[f]$ and $\nu_\partial[f]$ be defined as in (4). In case $D=R^N_+$, $\nu_\partial(x,\cdot)$ and $\nu_D(x,\cdot)$ are called *spatially homogeneous*, if there are measures $\tilde{\nu}_\partial(\cdot)$ on ∂D and $\tilde{\nu}_D(\cdot)$ on D such that, for each measurable subset A of $\bar{D}$,

$$\begin{aligned}&\nu_\partial(x,\{y+x\mid y\in A\})=\tilde{\nu}_\partial(A),\\ &\nu_D(x,\{y+x\mid y\in A\})=\tilde{\nu}_D(A),\qquad x\in\partial D.\end{aligned}\tag{6}$$

A spatially homogeneous $\nu_\partial(x,\cdot)$ is called *symmetric* with respect to x, if

$$\tilde{\nu}_\partial(\{-y\mid y\in A\})=\tilde{\nu}_\partial(A).$$

When $D=R^N_+$, let K_0 be the set of all real valued rapidly decreasing functions of class C^∞ on $\bar{D}$. When D is bounded, let $K_0=C^\infty(\bar{D})$. For f and g on $\bar{D}$, we write

$$(f,g)=\int_D f(x)\overline{g(x)}dx,\ \langle f,g\rangle=\int_{\partial D} f(x)\overline{g(x)}d\tilde{x},$$
$$\langle f,g\rangle_n=\sum_{|\alpha|\leq n}\langle\tilde{D}^\alpha f,\tilde{D}^\alpha g\rangle,$$

$\|f\|=(f,f)^{1/2}$, $\|f\|_\partial=\langle f,f\rangle^{1/2}$, $\|f\|_{\partial,n}=\langle f,f\rangle_n^{1/2}$, where $d\tilde{x}$ is the surface element of ∂D. In case $D=R^N_+$, we write, for $s>0$,

$$\langle f,g\rangle_s=\langle(1+|z|)^s\hat{f}(z),\ (1+|z|)^s\hat{g}(z)\rangle,\qquad \|f\|_{\partial,s}=\langle f,f\rangle_s^{1/2},$$

where $\hat{f}$ and $\hat{g}$ are the Fourier transforms of f and g as functions on ∂D.

Proposition 1. *Let* $\nu_D(f)(x)=\int_D \nu_D(x,dy)f(y)$ *be continuous in* $x\in\partial D$, *and let*

(7) $$\|\nu_D(f)\|_\partial \leq C\,\|f\|,\qquad f\in K_0.$$

Let $\nu_D(x,\cdot)$ *be spatially homogeneous when* $D=R^N_+$, *and let* $m=0$ *when* D *is bounded. Then, for* f *and* g *in* K_0, *and a constant* $\bar\nu$,

(8) $$|\langle \nu_D[f],g\rangle|\leq \bar\nu(\|f\|\cdot\|g\|_\partial+\|f\|_{\partial,m/2}\cdot\|g\|_{\partial,m/2}).$$

Proposition 2. *Let* $D=R^N_+$. *If* $\nu(x,\cdot)$ *satisfies* (5) *and* $\nu_\partial(x,\cdot)$ *is spatially homogeneous, then, for each* $\varepsilon>0$, *there is a constant* C_ε *such that*

(9) $$|\langle \nu_\partial[f],g\rangle|\leq \varepsilon\,\|f\|_{\partial,(m+1)/2}\cdot\|g\|_{\partial,(m+1)/2}+C_\varepsilon\,\|f\|_\partial\cdot\|g\|_\partial,\quad f,g\in K_0.$$

Proposition 3. *Let* $D=R^N_+$, *and* $m/2<s<(m+1)/2$. *We assume that*

(i) $\nu_\partial(x,\cdot)$ *is spatially homogeneous,*

(ii) *there are circular cones* $C^1,C^2,\cdots,C^l$ *on* ∂D *with the origin* $0=(0,\cdots,0)$ *as the vertices and with the vertical angles less than* $\pi/2$, *and* $\partial D=\bigcup_{1\leq k\leq l} C^k$,

(iii) *there is a sequence* $r_1=1,r_2,r_3,\cdots$ *such that*

$$\underline{a}\cdot r_j<r_{j+1}<\bar a\cdot r_j,\qquad for\ 0<\underline{a}<\bar a<1,\ j=1,2,\cdots$$
$$\underline{c}\cdot r_j^{-2s}<\tilde\nu(C^k_j)<\bar c\cdot r_j^{-2s},\qquad where\ C^k_j=\{x\in C^k\,|\,r_{j+1}<|x|\leq r_j\}.$$

Then, there are constants $\underline{b}$, *and* $\bar b$ *such that, for* f *and* g *in* K_0,

(10) $$-\langle \nu_\partial[f],f\rangle\geq \underline{b}\,\|f\|^2_{\partial,s},$$

(11) $$|\langle \nu_\partial[f],g\rangle|\leq \bar b\,\|f\|_{\partial,s}\cdot\|g\|_{\partial,s}.$$

3. We define, for f and g in K_0,

$$(f,g)_\delta=(f,g)+\langle f,g\cdot|\delta|\rangle,\qquad \|f\|_\delta=(f,f)_\delta^{1/2},$$
$$B_\lambda(f,g)=(\lambda f-Af,g)_\delta-\langle Lf,g\rangle,\qquad \lambda\geq 0.$$

Now, we consider the following four cases (I)–(IV).

(I) $D=R^N_+$. $\tilde A$ is strongly elliptic of order $2n$ uniformly on ∂D. $\nu(x,\cdot)$ satisfies (5), $\nu_\partial(x,\cdot)$ is spatially homogeneous and symmetric with respect to x, and $\nu_D(x,\cdot)$ vanishes.

(II) $D=R^N_+$, and $n\geq(m+1)/2$. $\tilde A$ is strongly elliptic of order $2n$ uniformly on ∂D. $\nu(x,\cdot)$ satisfies (5), $\nu_\partial(x,\cdot)$ is spatially homogeneous, and $\nu_D(x,\cdot)$ satisfies the condition of Proposition 1.

(III) $D=R^N_+$, and $n\leq m/2$. $\tilde A$ is a differential operator of order at most $2n$ on ∂D. $\nu(x,\cdot)$ satisfies (5), $\nu_D(x,\cdot)$ and $\nu_\partial(x,\cdot)$ satisfy the conditions of Propositions 1 and 3, respectively.

(IV) D is bounded, and $m=0$. $\tilde A$ is strongly elliptic of order $2n$ uniformly on ∂D. $\nu[f](x)$ is continuous for each C^∞ function f on $\bar D$. $\nu_\partial(x,\cdot)$ has a symmetric density $\nu_\partial(x,y)$ with respect to the surface element $d\tilde x$, and $\nu_D(x,\cdot)$ satisfies the condition of Proposition 1.

Proposition 4. *If one of the conditions* (I), (II), (III) *and* (IV) *is satisfied, then there is a constant* $\lambda_0>0$ *such that* $\|f\|_L=B_{\lambda_0}(f,f)^{1/2}$ *is a* *norm on* K_0, *and for* f *and* g *in* K_0,

(12) $$|B_\lambda(f,g)|\leq k_\lambda\|f\|_L\cdot\|g\|_L, \qquad \lambda\geq\lambda_0,$$

(13) $$B_\lambda(f,f)\geq k\|f\|_L^2, \qquad \lambda\geq\lambda_0,$$

(14) $$B_\lambda(f,f)\geq\|f\|_\delta^2, \qquad \lambda\geq\lambda_0.$$

When one of (I), (II), (III) and (IV) is satisfied, let H_δ and K be the completions of K_0 with respect to $\|\ \ \|_\delta$ and $\|\ \ \|_L$, respectively. Since it can be proved that $\|\ \ \|_L$ is closable with respect to $\|\ \ \|_\delta$, K is imbedded in H_δ uniquely. By Proposition 4, $B_\lambda(f,g)$ can be extended to a bilinear form on K, which, with the same notation, satisfies (12)–(14) for f,g in K. Moreover, it can be proved as in [7] that the following definition is possible.

Definition. Let $\mathfrak{D}(A_L)$ be the set of all f in K such that there are a sequence $\{f_n\}$ in K_0 and g in H_δ such that $\lim_{n\to\infty}\|f_n-f\|_L=0$, and

$$\lim_{n\to\infty}\{(Af_n-g,h)_\delta+\langle Lf_n,h\rangle\}=0, \qquad \text{for each } h\in K_0.$$

We define A_L by $A_Lf=g$ for above f in $\mathfrak{D}(A_L)$.

Theorem. *Under one of the conditions* (I), (II), (III) *and* (IV), A_L *is the generator of a semigroup* $\{T_t, t\geq 0\}$ *of linear operators on* H_δ, *which is strongly continuous in* $t\geq 0$, *and satisfies*

$$\|T_tf\|_\delta\leq e^{\lambda_0 t}\|f\|_\delta, \qquad f\in H_\delta.$$

The domain $\mathfrak{D}(A_L)$ *is a dense subspace of* K.

Remark. If f in K_0 satisfies $Lf(x)=0$ on ∂D, then f belongs to $\mathfrak{D}(A_L)$. Conversely, if f belongs to $\mathfrak{D}(A_L)\cap K_0$ and A_Lf is continuous on $\bar{D}$, then f satisfies $Lf(x)=0$ on ∂D.

The conditions in Proposition 3 are satisfied in the followings.

Example 1. $D=R_+^N$. $\tilde{\nu}_\partial(\cdot)$ in (6) vanishes on $\partial D\cap\{x\,|\,|x|>1\}$ and

$$\tilde{\nu}_\partial(dx)=|x|^{-2s-1}d\tilde{x}, \qquad |x|\leq 1.$$

Example 2. $D=R_+^N$ and $0<a<1$. $\underline{\underline{x}}_{k,j}$ and $-\underline{\underline{x}}_{k,j}$ are the points whose k-th coordinates are a^j and $-a^j$ respectively, and other coordinates are 0. $\tilde{\nu}_\partial(\cdot)$ is concentrated on $\{\underline{\underline{x}}_{k,j}, -\underline{\underline{x}}_{k,j}, i\leq k\leq N-1, j\geq 1\}$, and

$$\tilde{\nu}_\partial(\{\underline{\underline{x}}_{k,j}\})=\tilde{\nu}_\partial(\{-\underline{\underline{x}}_{k,j}\})=c_ka^{-2js}, \qquad 1\leq k\leq N-1,\ j\geq 1.$$

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