88. *The Range of Picard Dimensions**)

By Mitsuru NAKAI
Department of Mathematics, Nagoya Institute of Technology

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1. **Densities and Picard dimensions.** We will view the punctured unit disk $\Omega: 0<|z|<1$ as an *end* of $0<|z|\leqq+\infty$, a parabolic Riemann surface, so that the unit circle $|z|=1$ is the relative boundary $\partial\Omega$ of Ω and the origin $z=0$ is the single ideal boundary component $\delta\Omega$ of Ω. A *density* P on Ω is a nonnegative locally Hölder continuous function $P(z)$ on $\bar{\Omega}: 0<|z|\leqq 1$ which may or may not have a singularity at $\delta\Omega$. We denote by $PP(\Omega;\partial\Omega)$ the class of nonnegative solutions u of $\Delta u = Pu$ on Ω with vanishing boundary values on $\partial\Omega$. We also denote by $PP_1(\Omega;\partial\Omega)$ the subclass of $PP(\Omega;\partial\Omega)$ consisting of functions u with the normalization $u(a)=1$ for some fixed point a in Ω. We denote by ex. $PP_1(\Omega;\partial\Omega)$ the set of extreme points in the convex set $PP_1(\Omega;\partial\Omega)$. The cardinal number $\sharp(\text{ex.}\ PP_1(\Omega;\partial\Omega))$ of ex. $PP_1(\Omega;\partial\Omega)$ will be referred to as the *Picard dimension*, $\dim P$ in notation, of a density P at $\delta\Omega$:

(1) $$\dim P = \sharp(\text{ex.}\ PP_1(\Omega;\partial\Omega)).$$

It is easily seen (cf. e.g. [7]) that $\dim P\geqq 1$ for any density P on Ω. A density P on Ω with $\dim P=1$ is said to satisfy the *Picard principle* at $\delta\Omega$.

2. **Problem and result.** We denote by $\mathscr{D}(\Omega)$ the class of densities on Ω. Consider the mapping $\dim: \mathscr{D}(\Omega)\to\{\text{cardinal numbers}\}$ defined by $P\mapsto \dim P$. We proposed to study the *range* $\dim\mathscr{D}(\Omega)=\{\dim P; P\in\mathscr{D}(\Omega)\}$ of the mapping dim in our former paper (cf. [5]). Virtually nothing has been known on $\dim\mathscr{D}(\Omega)$ except for the following simple fact (cf. [4], [6], [2]):

$$\dim P_\lambda = \begin{cases} 1 & (\lambda\leqq 2) \\ \mathfrak{c} & (\lambda>2) \end{cases}$$

where P_λ is the density on Ω given by $P_\lambda(z)=|z|^{-\lambda}$ for real numbers λ and $\mathfrak{c}$ is the cardinal number of continuum. In view of this our *problem* is to determine whether the range $\dim\mathscr{D}(\Omega)$ contains cardinal numbers between 1 and $\mathfrak{c}$. Specifically we are interested in the question whether $\dim\mathscr{D}(\Omega)$ contains every countable cardinal numbers ξ, i.e. $\xi=n$, a positive integer, or $\xi=\mathfrak{a}$, the cardinal number of countably infinite set. The purpose of this note is to announce and also to give

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an outline of the proof of the following

Theorem. *There exists a density* P_ξ *on* Ω *for any countable cardinal number* ξ *such that* $\dim P_\xi = \xi$. *Therefore the range* $\dim \mathcal{D}(\Omega)$ *contains every positive integer* n, $\mathfrak{a}$, *and* $\mathfrak{c}$.

Actually the author has been studying the Picard principle motivated by the feeling that $\dim P$ is either 1 or $\mathfrak{c}$. Therefore the above result is rather surprising to the author. Once the situation turned out to be as mentioned above, the determination of those densities P with e.g. $\dim P=2$ is equally important to that of those densities with the Picard principle, and the study of Picard principle ought to take these new dimensions into account.

3. Relative harmonic dimensions. A sequence $\{K_n\}_1^\infty$ of continua K_n in Ω will be referred to as a *$\mathcal{K}$-sequence* in Ω if $K_n \cap K_m = \emptyset$ $(n \neq m)$, $W = \Omega - \bigcup_1^\infty K_n$ is connected, and $\{K_n\}$ converges to $\delta\Omega$. We denote by $\mathcal{K}(\Omega)$ the set of $\mathcal{K}$-sequences in Ω. The relative boundary ∂W of the region $W = \Omega - \bigcup_1^\infty K_n$ for a $\mathcal{K}$-sequence $\{K_n\}_1^\infty$ is $\partial W = (\partial\Omega) \cup (\bigcup_1^\infty \partial K_n)$. We then consider the class $HP(W; \partial W)$ of nonnegative harmonic functions on W with vanishing boundary values on ∂W and the subclass $HP_1(W; \partial W)$ of $HP(W; \partial W)$ consisting of those functions with the normalization $u(a)=1$ for some fixed point a in W. Similarly to the Picard dimension we define the *relative harmonic dimension*, $\dim\{K_n\}$ in notation, of a $\mathcal{K}$-sequence $\{K_n\}$ at $\delta\Omega$ by

$$(2) \qquad \dim\{K_n\} = \#(\text{ex. } HP_1(W; \partial W)).$$

It is easy to see that $\dim\{K_n\} \geqq 1$ for any $\mathcal{K}$-sequence $\{K_n\}$ in Ω. We will see (cf. the last part of no. 5 below) that the range $\dim \mathcal{K}(\Omega)$ of the mapping $\dim : \mathcal{K}(\Omega) \to \{\text{cardinal numbers}\}$ also contains every countable cardinal number. The question of the range of relative harmonic dimensions formally resembles to that of harmonic dimensions of ends of infinite genus studied by Heins, Kuramochi, Ozawa, Constantinescu-Cornea, and others. We have positively settled the question whether $\mathfrak{c}$ belongs to $\dim \mathcal{K}(\Omega)$ or not. The question seems to have a close bearing with the Kobe mapping problem onto the circle regions.

4. Canonically associated densities. Suppose that each continuum $\overline{Y}_n$ in a $\mathcal{K}$-sequence $\{\overline{Y}_n\}$ in Ω is the closure of a Jordan region Y_n in Ω $(n=1,2,\cdots)$. Such a $\mathcal{K}$-sequence will be refferred to as a *$\mathcal{Y}$-sequence* in Ω and we denote by $\mathcal{Y}(\Omega)$ the class of $\mathcal{Y}$-sequences in Ω so that $\mathcal{Y}(\Omega) \subset \mathcal{K}(\Omega)$. Consider the region $W = \Omega - \bigcup_1^\infty \overline{Y}_n$ for a $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$ and a density P on Ω such that supp. $P \subset \bigcup_1^\infty Y_n = \Omega - \overline{W}$. We denote by H_u^W for each u in $PP(\Omega; \partial\Omega)$ the least nonnegative harmonic function on W with boundary values u on ∂W (cf. e.g. [1]). It is the lower envelope of the family of superharmonic functions s on W with the lower limit boundary values of s on ∂W is not less than

$u|\partial W$. Then the function $T_P u = u - H_u^W$ belongs to the class $HP(W;\partial W)$ for every u in $PP(\Omega;\partial\Omega)$, and $u \mapsto T_P u$ defines a mapping $T_P : PP(\Omega : \partial\Omega) \to HP(W;\partial W)$. It is easy to see that the mapping T_P is order-preserving (i.e. $u_1 \leqq u_2$ implies $T_P u_1 \leqq T_P u_2$), positively homogeneous (i.e. $T_P(\lambda u) = \lambda T_P u$ for nonnegative real numbers λ), and additive (i.e. $T_P(u_1 + u_2) = T_P u_1 + T_P u_2$). In general T_P may or may not be injective and similarly surjective. If the mapping T_P happens to be bijective, then the density P is said to be *canonically associated* with the $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$. If a density P on Ω is canonically associated with a $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$, then it is easy to see that

$$\dim P = \dim \{\overline{Y}_n\}. \tag{3}$$

5. Outline of the proof. We denote by P_f^U the solution of $\Delta u = Pu$ on an analytic Jordan region U with $\overline{U} \subset \Omega$ with boundary values f on ∂U where P is a density on Ω and f is in $C(\partial U)$. Given any Jordan region Y with $\overline{Y} \subset U$ and any positive number ε, there exists a density $P = P_{U,Y,\varepsilon}$ on Ω with supp. $P \subset Y$ and

$$\sup_Y |P_f^U| \leqq \varepsilon \int_{\partial U} |f|\, d\omega_U \tag{4}$$

for any f in $C(\partial U)$, where $d\omega_U$ is the harmonic measure on ∂U with respect to U evaluated at a fixed point in U. We only have to choose P sufficiently large in Y so that it is qualified as P in the above assertion. The technical detail for the construction is similar to that employed in [3] in a similar but slightly different setting. Let $\{\overline{Y}_n\}$ be any $\mathcal{Y}$-sequence on Ω and U_n be a slightly larger analytic Jordan region in Ω than Y_n containing $\overline{Y}_n$ $(n=1,2,\cdots)$. Using the densities P_n satisfying (4) for $U = U_n$, $Y = Y_n$, and $\varepsilon = \varepsilon_n$ $(n=1,2,\cdots)$, we construct a density P on Ω by $\sum_n P_n$. If the sequence $\{\varepsilon_n\}$ is chosen to converge to zero sufficiently rapidly, then it is seen that the sequence $\{\sup_{Y_n} u\}$ converges to zero for any u in $PP(\Omega;\partial\Omega)$. From this it follows that T_P is bijective and therefore P is canonically associated with $\{\overline{Y}_n\}$. In view of this and (3), the proof of the theorem is reduced to showing that $\dim \mathcal{Y}(\Omega)$ contains any countable cardinal number. In passing we remark that $\mathcal{Y}(\Omega) \subset \mathcal{K}(\Omega)$ implies that $\dim \mathcal{K}(\Omega)$ contains any countable cardinal number along with $\dim \mathcal{Y}(\Omega)$. Thus the proof of the theorem will be complete if we show an example of $\mathcal{Y}$-sequence $\{\overline{Y}_n\}$ with $\dim \{\overline{Y}_n\} = m$, any positive integer, or $\mathfrak{a}$.

6. Example 1. First we exhibit an example of a $\mathcal{Y}$-sequence $\{\overline{Y}_n\}_1^\infty$ with $\dim \{\overline{Y}_n\} = m$ for any given positive integer $m = 1, 2, \cdots$. Fix a sequence $\{a_\mu\}_1^\infty$ in $(0,1)$ with $a_{\mu+1} < a_\mu$ $(\mu = 1, 2, \cdots)$ and $\lim_{\mu\to\infty} a_\mu = 0$. We choose a sequence $\{b_\mu\}_1^\infty$ in $(0,1)$ with $a_{\mu+1} < b_\mu < a_\mu$ $(\mu = 1, 2, \cdots)$. Let $\theta_\nu = 2\pi(\nu - 1)/m$ $(\nu = 1, 2, \cdots, m)$, η be in $(0, \pi/m)$, and

$$S_{\mu\nu} = \{b_\mu < |z| < a_\mu,\ |\arg z - \theta_\nu| < \eta\}\ .$$

Observe that any positive integer n has a unique expression $n = (\mu - 1)m$

$+(\nu-1)$ with positive integers μ and ν with $1\leqq\nu\leqq m$. We set

$$Y_n=S_{\mu\nu} \qquad (n=(\mu-1)m+(\nu-1)).$$

Then the sequence $\{\bar{Y}_n\}_{n=1}^\infty=\{\bar{S}_{\mu\nu}\}$ $(\nu=1,2,\cdots,m;\mu=1,2,\cdots)$ is clearly a $\mathcal{Y}$-sequence. If we choose the sequence $\{b_\mu\}_1^\infty$ so as to make the sequence $\{b_\mu-a_{\mu+1}\}_{\mu=1}^\infty$ convergent to zero 'enough' rapidly, then we can show that $\dim\{\bar{S}_{\mu\nu}\}=m$. If $\delta\Omega$ is irregular for the Dirichlet problem for the region $\Omega-\bigcup\bar{S}_{\mu\nu}$ (i.e. if $\{b_\mu\}$ is so chosen), then it is well known that $\dim\{\bar{S}_{\mu\nu}\}=1$. We suspect that $\dim\{\bar{S}_{\mu\nu}\}$ is either 1 or m no matter how we choose $\{b_\mu\}$ but we are unable to prove it at present.

7. Example 2. Next we exhibit an example of $\mathcal{Y}$-sequence $\{\bar{Y}_k\}_{k=1}^\infty$ with $\dim\{\bar{Y}_k\}=\mathfrak{a}$. Fix again a sequence $\{a_m\}_1^\infty$ in $(0,1)$ with $a_{m+1}<a_m$ $(m=1,2,\cdots)$ and $\lim_{m\to\infty}a_m=0$. We choose a sequence $\{b_m\}_1^\infty$ in $(0,1)$ with $a_{m+1}<b_m<a_m$ $(m=1,2,\cdots)$ in a suitable fashion. We also fix sequences $\{\sigma_n\}$ and $\{\tau_n\}$ in $[0,2\pi)$ such that $\sigma_1=0$, $\sigma_n<\tau_n<\sigma_{n+1}$ $(n=1,2,\cdots)$, and $\lim_{n\to\infty}\sigma_n=2\pi$. Let

$$S_{mn}=\{b_m<|z|<a_m,\ \sigma_n<\arg z<\tau_n\} \qquad (m\geqq n).$$

Observe that any positive integer k has a unique expression $k=m(m-1)/2+n$ with positive integers m and n satisfying $m\geqq n$. Then we set

$$Y_k=S_{mn} \qquad (k=m(m-1)/2+n).$$

The sequence $\{\bar{Y}_k\}_{k=1}^\infty=\{\bar{S}_{mn}\}$ $(m\geqq n;\ m,\ n=1,2,\cdots)$ is clearly a $\mathcal{Y}$-sequence. We choose $\{b_m\}$ so as to make the sequence $\{b_m-a_{m+1}\}_{m=1}^\infty$ convergent to zero 'enough' rapidly, then we can show that $\dim\{\bar{S}_{mn}\}=\mathfrak{a}$.

In verifying that the examples 1 and 2 are required ones, the following simple fact (cf. e.g. [1]) plays an important role. Let $\{U_n\}_1^\infty$ be a sequence of disks U_n with $\bar{U}_n\subset\Omega$ such that $\bar{U}_n\cap\bar{U}_m=\emptyset$ $(n\neq m)$ and $\{\bar{U}_n\}$ converges to $\delta\Omega$. Consider a region $W=\Omega-\bigcup_1^\infty K_n$ for a $\mathcal{K}$-sequence $\{K_n\}$ in Ω. Set $V_n=W\cap U_n$ and $V=\bigcup_1^\infty V_n$. Let q be a Martin boundary point of the region W lying over $\delta\Omega$. If q does not belong to the closure $(W-V)^a$ of $W-V$ considered in the Martin compactification of W, then q is not a minimal point.

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