86. On the Smoothness of Infinitely Divisible Distributions Corresponding to Some Ordinary Differential Equations

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1. Introduction. In the course of the investigation of the limit theorems of the decomposable Galton-Watson processes, the author [1] has found a class of the infinitely divisible distributions closely related to the following Riccati equations.

Let

$$\phi(t)=\sum_{n=0}^{\infty} a_n t^n,\ B>0 \quad \text{and} \quad m\geqq 0 \tag{1.1}$$

be given. We assume that every $a_n\geqq 0$ and $\phi(t)$ converges for all t. Let $\psi(t,\lambda)$, $t\geqq 0$, be the solution of

$$\frac{d}{dt}\psi(t,\lambda)=-B\psi(t,\lambda)^2+\phi(t)\lambda, \qquad \psi(0,\lambda)=m\lambda, \tag{1.2}$$

with $\lambda\geqq 0$ being a parameter.

Then we have

Theorem 1. (i) *For each $t>0$, there exists a probability measure P_t on $[0,\infty)$ such that*

$$\int_0^\infty e^{-\lambda x}P_t(dx)=\exp\left\{-\int_0^t \psi(s,\lambda)ds\right\}. \tag{1.3}$$

(ii) *P_t is infinitely divisible.*

(iii) *The Lévy measure n_t of P_t has the finite moments of all order.*

The probabilistic proof of (i) will be given in a forthcoming paper [1]. An alternative proof, which can be applied to more general equations, was given by T. Watanabe [2]. If we assume (i), (ii) is easily seen from $a\psi(t,\lambda;\phi,B,m)=\psi(t,\lambda;a\phi,a^{-1}B,am)$ for any $a>0$. (iii) follows from the fact that $\psi(t,\lambda)$ is C^∞ at $\lambda=0$.

The purpose of this paper is to show the following

Theorem 2. *Suppose that $\sum_{n=0}^{\infty} a_n>0$. Then there exists $d(t)>0$ such that*

$$\left|\int_0^\infty e^{i\lambda x}P_t(dx)\right|\leqq\exp\{-d(t)\sqrt{|\lambda|}\}, \tag{1.4}$$

for all sufficiently large $|\lambda|$. Therefore P_t is absolutely continuous with respect to the Lebesgue measure and the density belongs to $C^\infty(\boldsymbol{R})$.

Remark. If $\sum_{n=0}^{\infty} a_n=0$ and $m>0$, it is easily seen that P_t is a gamma distribution and the density belongs to $C^\infty(\boldsymbol{R}-\{0\})$.

2. Proof of Theorem 2. We first state a lemma which will be shown in § 3.

Lemma 2.1.

$$\lim_{\lambda\to\infty}(\sqrt{\lambda})^{-1}\int_0^t \psi(s,\lambda)ds=\int_0^t \sqrt{B^{-1}\phi(s)}ds>0, \qquad t>0. \tag{2.1}$$

Without loss of generality we assume that $t=1$. By Theorem 1, there exists $c\geqq 0$ and a measure $n(dy)$ on $[0,\infty)$ with $n(\{0\})=0$ such that

$$\int_0^1 \psi(s,\lambda)ds=c\lambda+\int_0^\infty (1-e^{-\lambda y})n(dy).$$

But by (2.1), we have $c=0$ and so

$$\int_0^1 \psi(s,\lambda)ds=\int_0^\infty (1-e^{-\lambda y})n(dy)=\lambda\int_0^\infty e^{-\lambda y}n(y)dy, \tag{2.2}$$

where $n(y)=n((y,\infty))$. Hence by (2.1), we have

$$\lim_{\lambda\to\infty}\sqrt{\lambda}\int_0^\infty e^{-\lambda y}n(y)dy=\int_0^1 \sqrt{B^{-1}\phi(s)}ds\equiv A_1>0. \tag{2.3}$$

Therefore by Theorem 4.3 in [3, p. 192],

$$\lim_{y\to 0}(\sqrt{y})^{-1}\int_0^y n(z)dz=\Gamma\left(\frac{3}{2}\right)^{-1}A_1\equiv A_2>0. \tag{2.4}$$

Since $2^{-1}yn(2^{-1}y)\geqq\int_{2^{-1}y}^y n(z)dz\geqq 2^{-1}yn(y)$, we have by (2.4),

$$4A_2>\overline{\lim_{y\to 0}}\sqrt{y}\,n(y)\geqq\underline{\lim_{y\to 0}}\sqrt{y}\,n(y)>2^{-1}(\sqrt{2}-1)A_2\equiv A_3>0. \tag{2.5}$$

Take $\sqrt{A_4}<4^{-1}A_2^{-1}A_3$, then it follows from (2.4) and (2.5) that

$$\begin{aligned}\int_0^y z^2n(dz)&=\int_0^y 2z(n(z)-n(y))dz\geqq\int_0^{A_4y} 2z(n(A_4y)-n(y))dz \\ &=A_4^2y^2(n(A_4y)-n(y))\geqq A_4^2y^2(A_3(\sqrt{A_4y})^{-1}-4A_2(\sqrt{y})^{-1}) \\ &\equiv A_5\sqrt{y^3},\end{aligned} \tag{2.6}$$

for all sufficiently small y. Therefore we have

$$\begin{aligned}\left|\int_0^\infty e^{i\lambda y}P(dx)\right|&=\exp\left\{-\int_0^\infty (1-\cos(\lambda y))n(dy)\right\} \\ &\leqq\exp\left\{-\int_0^{|\lambda|^{-1}} 4^{-1}\lambda^2y^2n(dy)\right\}\leqq\exp\{-4^{-1}A_5\sqrt{|\lambda|}\},\end{aligned} \tag{2.7}$$

for all sufficiently large $|\lambda|$.

3. Proof of Lemma 2.1. In this section, $\psi_m(t,\lambda)$ denotes the unique solution of

$$\frac{d}{dt}\psi_m(t,\lambda)=-B\psi_m(t,\lambda)^2+\phi(t)\lambda, \qquad \psi_m(0,\lambda)=m\lambda. \tag{3.1}$$

Proposition 3.1.

$$0\leqq\psi_0(t,\lambda)\leqq\sqrt{B^{-1}\phi(t)\lambda}, \qquad t\geqq 0. \tag{3.2}$$

$$\lim_{\lambda\to\infty}(\sqrt{\lambda})^{-1}\psi_0(t,\lambda)=\sqrt{B^{-1}\phi(t)}, \qquad t>0. \tag{3.3}$$

The convergence in (3.3) *is monotone.*

$$0 \leqq \psi_m(t,\lambda) - \psi_0(t,\lambda) \leqq \frac{m\lambda}{1+tBm\lambda}, \qquad t \geqq 0. \tag{3.4}$$

We first prove Lemma 2.1, assuming Proposition 3.1. If $m=0$, then (2.1) follows from (3.2) and (3.3). If $m>0$, then (2.1) follows from the result of the case $m=0$ and (3.4).

We now proceed to the proof of Proposition 3.1. By (3.1), $\psi_0(t,\lambda)=\int_0^t \lambda\phi(s)\exp\left\{-\int_s^t B\psi_0(r,\lambda)dr\right\}ds \geqq 0$. If there exists $T>0$ such that $\psi_0(T,\lambda)>\sqrt{B^{-1}\phi(T)\lambda}$, set $t_0=\sup\{t<T;\ \psi_0(t,\lambda)\leqq\sqrt{B^{-1}\phi(t)\lambda}\}$. Then we get a contradiction;

$$\begin{aligned}\psi_0(T,\lambda) &= \psi_0(t_0,\lambda)+\int_{t_0}^T (-B\psi_0(t,\lambda)^2+\phi(t)\lambda)dt \\ &\leqq \psi_0(t_0,\lambda) \leqq \sqrt{B^{-1}\phi(t_0)\lambda} \leqq \sqrt{B^{-1}\phi(T)\lambda}.\end{aligned}$$

Next we shall show (3.3). Set

$$\theta(t,\lambda)=(\sqrt{\lambda})^{-1}\psi_0(t,\lambda). \tag{3.5}$$

By (3.2), we have

$$0 \leqq \theta(t,\lambda) \leqq \sqrt{B^{-1}\phi(t)}. \tag{3.6}$$

$\theta(t,\lambda)$ satisfies

$$\begin{cases} \dfrac{d}{dt}\theta(t,\lambda)=\sqrt{\lambda}(-B\theta(t,\lambda)^2+\phi(t)) \geqq 0, \\ \theta(0,\lambda)=0. \end{cases} \tag{3.7}$$

Differentiating with respect to λ,

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\partial\theta}{\partial\lambda}(t,\lambda) = -2B\sqrt{\lambda}\,\theta(t,\lambda)\dfrac{\partial\theta}{\partial\lambda}(t,\lambda)+(2\sqrt{\lambda})^{-1}(-B\theta(t,\lambda)^2+\phi(t)), \\ \dfrac{\partial\theta}{\partial\lambda}(0,\lambda)=0. \end{cases}$$

Since $-B\theta(t,\lambda)^2+\phi(t)\geqq 0$ by (3.6), $\dfrac{\partial\theta}{\partial\lambda}(t,\lambda)\geqq 0$ and hence $\theta(t,\lambda)$ is increasing in λ. If we set $\eta(t)=\lim_{\lambda\to\infty}\theta(t,\lambda)$, then by (3.2) and (3.7), we have

$$\begin{aligned} 0 &= \lim_{\lambda\to\infty}\lambda^{-1}\psi_0(t,\lambda)=\lim_{\lambda\to\infty}(\sqrt{\lambda})^{-1}\theta(t,\lambda) \\ &= \lim_{\lambda\to\infty}\int_0^t(-B\theta(s,\lambda)^2+\phi(s))ds=\int_0^t(-B\eta(s)^2+\phi(s))ds. \end{aligned} \tag{3.8}$$

Therefore we have

$$\eta(t)=\sqrt{B^{-1}\phi(t)} \qquad \text{a.e. } t. \tag{3.9}$$

Since both sides in (3.9) are increasing and the right side is continuous, (3.9) holds for all $t>0$. This completes the proof of (3.3). By the uniqueness of the solution of (3.1) we have $\psi_m(t,\lambda)\geqq\psi_0(t,\lambda)$. Set $\xi(t)=\psi_m(t,\lambda)-\psi_0(t,\lambda)$. Then by (3.1),

$$\begin{cases} \dfrac{d\xi}{dt}(t) = -B(\psi_m(t,\lambda)-\psi_0(t,\lambda))(\psi_m(t,\lambda)+\psi_0(t,\lambda)) \leqq -B\xi(t)^2, \\ \xi(0)=m\lambda, \end{cases}$$

which implies (3.4).

References

[1] S. Sugitani: On the limit distributions of decomposable Galton-Watson processes. Proc. Japan Acad., **55**A, 334–336 (1976).

[2] T. Watanabe: Infinitely divisible distributions and ordinary differential equations. Ibid., **55**A, 375–378 (1979).

[3] D. V. Widder: The Laplace Transform. Princeton University Press, Princeton, New Jersey (1946).