8. *L-Equivalence Classes of Submanifolds in Complex Projective Spaces*

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1. Introduction. In a celebrated paper [7], Thom has developed a classification theory of submanifolds of a compact differentiable manifold M roughly as follows. Two oriented submanifolds N_1 and N_2 of M of codimension p are said to be *L-equivalent* if they are oriented cobordant in $M\times I$. Let $\mathcal{L}_p(M)$ be the set of L-equivalence classes of submanifolds of M of codimenision p. Then by making use of the transversality theorem, he has established a bijection

$$\pi:\mathcal{L}_p(M)\xrightarrow{\sim}[M, MSO(p)]$$

where the right hand side stands for the set of homotopy classes of maps from M to $MSO(p)$, the Thom space for the group $SO(p)$. The correspondence is given by the so-called Pontrjagin-Thom map $\pi_N: M\to MSO(p)$, which is defined for every oriented submanifold N of M of codimension p. If we consider only those submanifolds of M with complex normal bundles, we still have a bijection

$$\pi:\mathcal{L}_p^C(M)\xrightarrow{\sim}[M, MU(p)]$$

where $\mathcal{L}_p^C(M)$ is the set of (suitably modified) L-equivalence classes of submanifolds of codimension $2p$ with complex normal bundles of M and $MU(p)$ is the Thom space for the group $U(p)$. Now assume that M is an n-dimensional compact complex manifold. Then a natural question arises:

Question. Which element of $\mathcal{L}_p^C(M)$ or $\mathcal{L}_{2p}(M)$ can be represented by a *complex* submanifold of M?

If M is a compact Kähler manifold, then there are some obvious conditions for an element in $\mathcal{L}_p^C(M)$ to be represented by a complex submanifold N of codimension p coming from the facts that the Poincaré dual of N is a non zero element of $H^{p,p}(M)$ and also, under the Gysin homomorphism, the Chern classes of the normal bundle of N go to the set of cohomology classes of type (q, q) in the Hodge decomposition of the complex cohomologies of M. In this note we formulate a general condition other than the above and show that it is actually satisfied for a particular case when the ambiant manifold is the complex projective

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n-space $P_n(C)$. The precise statement of the condition will be given in § 2 in terms of the extended de Rham theory due to Sullivan. In fact the present work has been motivated by his work [6].

The details of the results will appear elsewhere.

2. Statement of result. We begin by recalling several facts from the theory of Sullivan (see [5], [6], [3]). Thus let X be a simply connected triangulated space and let $\mathcal{E}(X)$ be the differential graded algebra of $\boldsymbol{Q}$-polynomial forms on X. A differential graded algebra $\mathcal{A}$ will be called a *model* for X (or $\mathcal{E}(X)$) if there is a map (in the sense of differential graded algebras) $\phi: \mathcal{A} \to \mathcal{E}(X)$ such that ϕ induces an isomorphism on cohomology. Among the models, there exists a particular model $\rho_X: \mathcal{M}(X) \to \mathcal{E}(X)$, called the *minimal model* of X, which is essentially equivalent to the rational Postnikov tower of X. A space X will be called *formal over* $\boldsymbol{Q}$ or simply *formal* if there is a map of differential graded algebras $\psi_X: \mathcal{M}(X) \to H^*(X;\boldsymbol{Q})$ inducing the identity on cohomology. Similarly a continuous map $f: X \to Y$ between two formal spaces X and Y is said to be *formal* if the following diagram is homotopy commutative

$$\begin{array}{ccc} \mathcal{M}(Y) & \xrightarrow{\hat{f}} & \mathcal{M}(X) \\ {\scriptstyle \psi_Y}\downarrow & & \downarrow{\scriptstyle \psi_X} \\ H^*(Y;\boldsymbol{Q}) & \xrightarrow[f^*]{} & H^*(X;\boldsymbol{Q}) \end{array}$$

where $\hat{f}$ is the induced map of f and ψ_X, ψ_Y are the maps defining the formalities of X and Y. The rational homotopy theory of a formal map is determined by the induced homomorphism on cohomology. Now the main result of [3] (see also [6] for the statement over $\boldsymbol{Q}$) is the following

Theorem 1 (Deligne, Griffiths, Morgan, Sullivan). *Simply connected compact Kähler manifolds and holomorphic maps between them are formal.*

Now we go back to our problem. Let M be a simply connected compact Kähler manifold and let N be a complex submanifold of M of codimension p. We have the Pontrjagin-Thom map $\pi_N: M \to MU(p)$ (or $MSO(2p)$). It is known by Theorem 1 that M is a formal space and also it can be easily seen that Thom spaces $MU(p)$ and $MSO(2p)$ are formal (cf. [6]. It also follows from Proposition 2 below.). Under these situations, the following statement could be taken for one of the precise forms of our *Question*.

Question′. Is the Pontrjagin-Thom map $\pi_N: M \to MU(p)$ (or $MSO(2p)$) formal?

In the stable range $p \geqq \frac{1}{2}(n+1)$, every map $f: M \to MU(p)$ (or

$MSO(2p)$) is formal because in this range $MU(p)$ (or $MSO(2p)$) can be considered as a product of Eilenberg-Maclane spaces over $\boldsymbol{Q}$. However beyond that range the spaces $MU(p)$ and $MSO(2p)$ are far from being products of Eilenberg-Maclane spaces (except the case $p=1$) and if *Question′* would be answered affirmatively for a manifold M, then it should imply that there exist topological obstructions for an element in $\mathcal{L}_p^C(M)$ to be represented by a complex submanifold. At present, we have no general confidence about the above *Question′*. Here we only claim that it holds affirmatively for the particular case when $M=P_n(\boldsymbol{C})$. Namely we have

Theorem 2. *Let N be a complex submanifold of $P_n(\boldsymbol{C})$ of codimension P. Then the Pontrjagin-Thom map $\pi_N: P_n(\boldsymbol{C})\to MU(p)$ is formal.*

Besides the one stated before this theorem, it has another meaning that the L-equivalence class of a complex submanifold of $P_n(\boldsymbol{C})$ is determined up to finite number of possibilities or exactly over $\boldsymbol{Q}$ by the cohomology invariants. This point has the following

Corollary 3. *Let N be a nonsingular subvariety of $P_n(\boldsymbol{C})$. Then the set of those L-equivalence classes which can be represented by conjugate varieties of N is finite.*

3. Preliminaries. Let $f: X\to Y$ be a continuous map. Then we have the cofibration $X\xrightarrow{f}Y\xrightarrow{p}C_f$ where C_f is the cofibre (or the mapping cone) of f and p is the natural inclusion. Now we give a similar construction in the category of differential graded algebras and their maps. Thus let $\phi: \mathcal{B}\to\mathcal{A}$ be a map of differential graded algebras. Then we define a differential graded algebra C_ϕ, called the *cofibre* of ϕ, by setting $C_\phi=\oplus_q C^q$, $C^q=\{(\sum\alpha_i t^i, \sum\bar{\alpha}_j t^j, \beta)\,;\, \alpha_i\in\mathcal{A}^{q-1}, \bar{\alpha}_j\in\mathcal{A}^q, \beta\in\mathcal{B}^q$ and $\phi(\beta)=\sum\bar{\alpha}_j\}$. We can define canonical differential and multiplication in C_ϕ so that it becomes a differential graded algebra. The natural sequence $C_\phi\xrightarrow{\iota}\mathcal{B}\xrightarrow{\phi}\mathcal{A}$ defines a long exact sequence of cohomology groups. Now let $f: X\to Y$ be a continuous map and suppose that there are given models $\mathcal{A}\to\mathcal{E}(X)$, $\mathcal{B}\to\mathcal{E}(Y)$ and a map $\phi:\mathcal{B}\to\mathcal{A}$ making the following diagram commutative

$$\begin{array}{ccc}\mathcal{B} & \xrightarrow{\phi} & \mathcal{A}\\ \downarrow & & \downarrow\\ \mathcal{E}(Y) & \xrightarrow{f^*} & \mathcal{E}(X).\end{array}$$

Then we have

Proposition 1. *C_ϕ is a model for C_f. Namely there is a map $C_\phi\to\mathcal{E}(C_f)$ inducing an isomorphism on cohomology.*

Let Y be a formal space and let $f: X\to Y$ be a continuous map. Let $K=\operatorname{Ker}(f^*: H^*(Y;\boldsymbol{Q})\to H^*(X;\boldsymbol{Q}))$ and $I=H^*(Y;\boldsymbol{Q})/K$. Since K

is an ideal of $H^*(Y;\boldsymbol{Q})$, I has a structure of an algebra. We write $F(K)$ and $F(I)$ for the formal spaces (=formal rational homotopy types) corresponding to the algebras K and I respectively. Let $Y_{(0)}$ be the rational homotopy type of Y and let $F(I)\xrightarrow{i}Y_{(0)}$, $Y_{(0)}\xrightarrow{k}F(K)$ be the formal maps corresponding to the projection $H^*(Y;\boldsymbol{Q})\to I$ and the inclusion $K\subset H^*(Y;\boldsymbol{Q})$. Then the sequence $F(I)\to Y_{(0)}\to F(K)$ is a cofibration. Now we make the following

Definition. Let Y be a formal space. A continuous map $f:X\to Y$ is called *pseudo formal* if the map $X\xrightarrow{f}Y\to Y_{(0)}$ lifts to $F(I)$.

It can be shown that formal map is pseudo formal. The following is the main technical result.

Proposition 2. *Let Y be a formal space and let $f:X\to Y$ be a pseudo formal map. Then the cofibre C_f has the rational homotopy type of $F(K)\vee S(\mathrm{Cok}\,f^*)$, where $K=\mathrm{Ker}\,(f^*:H^*(Y;\boldsymbol{Q})\to H^*(X;\boldsymbol{Q}))$ and $S(\mathrm{Cok}\,f^*)=\vee(\underbrace{S^{q+1}\vee\cdots\vee S^{q+1}}_{d(q)\text{-times}})$, $d(q)=\dim\mathrm{Cok}\,(f^*:H^q(Y;\boldsymbol{Q})\to H^q(X;\boldsymbol{Q}))$. Moreover the natural map $p:Y\to C_f$ is rationally homotopic to the projection onto the factor $F(K)$ by the map k. In particular the space C_f and the map p are formal.*

This is proved by using Proposition 1. From Proposition 2, we can deduce

Proposition 3. *Let N be a differentiable submanifold of $P_n(\boldsymbol{C})$ of codimension $2p$ and let $\pi:P_n(\boldsymbol{C})\to T(\nu)$ be the collapsing map onto $T(\nu)$, the Thom space of the normal bundle of N. Assume that N is not homologous to zero in $P_n(\boldsymbol{C})$. Then $T(\nu)$ has the rational homotopy type of $P_n(\boldsymbol{C})/P_{p-1}(\boldsymbol{C})\vee S(\mathrm{Cok}\,j^*)$, where $j:U\to P_n(\boldsymbol{C})$ is the inclusion, $U=P_n(\boldsymbol{C})-N$. Moreover the map π is rationally homotopic to the projection onto the factor $P_n(\boldsymbol{C})/P_{p-1}(\boldsymbol{C})$, hence it is formal.*

We need also the following

Proposition 4. *Let X be a formal space and let ξ be a p-dimensional complex vector bundle (or $2p$-dimensional oriented real vector bundle) over X. Then the Thom space $T(\xi)$ and the natural map $T(\xi)\to MU(p)$ (or $MSO(2p)$) are formal.*

4. Sketch of proof. The proof of Theorem 2 goes roughly as follows. First it is easy to show that every map $f:P_n(\boldsymbol{C})\to MU(p)$ is formal if $p\geqq\frac{1}{3}(n-2)$. So let N be a complex submanifold of $P_n(\boldsymbol{C})$ of codimension $p<\frac{1}{3}(n-2)$ $\left(\text{thus }\dim N\geqq\frac{2}{3}(n+1)\right)$. Then by a result of Barth and Larsen [2], N is simply connected and hence it is a formal space by Theorem 1. Now the Pontrjagin-Thom map π_N factors through $T(\nu)$; $\pi_N:P_n(\boldsymbol{C})\xrightarrow{\pi}T(\nu)\to MU(p)$. But by Propositions 3 and

4 the maps π and $T(\nu)\to MU(p)$ are formal. Hence π_N is formal.

Remark. (i) There is a proof of Theorem 2 based on a result of Barth [1] concerning the cohomology of complex submanifolds of $P_n(C)$. However the proof sketched above is more intrinsic.

(ii) Perhaps we should mention a conjecture of Hartshorne which says that the types of complex submanifolds of $P_n(C)$ of small codimensions would be very limited. (See [4], the precise statement is this: if N is a complex submanifold of $P_n(C)$ of dimension $>\frac{2}{3}n$, then N is a complete intersection.)

References

[1] W. Barth: Transplanting cohomology classes in complex-projective space. Amer. J. Math., **92**, 951–967 (1970).

[2] W. Barth and M. F. Larsen: On the homotopy groups of complex projective algebraic manifolds. Math. Scand., **30**, 88–94 (1972).

[3] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan: Real homotopy theory of Kähler manifolds. Invent. Math., **29**, 245–274 (1975).

[4] R. Hartshorne: Varieties of small codimension in projective space. Bull. Amer. Math. Soc., **80**, 1017–1032 (1974).

[5] D. Sullivan: Differential forms and the topology of manifolds. Manifolds-Tokyo 1973, Proc. of Intern. Conf. on Manifolds and related topics in topology, Univ. of Tokyo Press, 37–49 (1975).

[6] ——: Infinitesimal computations in topology. Publ. Math. IHES, **47**, 269–331 (1977).

[7] R. Thom: Quelques propriétés globales des variétés différentiables. Comment. Math. Helv., **28**, 17–86 (1954).