69. On Sufficient Conditions for the Boundedness of Pseudo-Differential Operators

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We report here that pseudo-differential operators are bounded in L_p, $1<p<\infty$, if some considerably weak conditions on the smoothness of their symbols are satisfied.

1. Notations. If $x=(x_1, \cdots, x_n)$ is a point in the n-dimensional Euclidean space $\boldsymbol{R}^n$, and $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index, then we write $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $\partial_{x_j}=\partial/\partial x_j$, $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$, $\langle x\rangle=(1+|x|^2)^{1/2}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$. We denote by Δ the difference operator, and adopt the following conventions:

$$\Delta_y a(x, \xi, x')=a(x+y, \xi, x')-a(x, \xi, x'),$$
$$\Delta_\eta a(x, \xi, x')=a(x, \xi+\eta, x')-a(x, \xi, x'),$$
$$\Delta_{y'} a(x, \xi, x')=a(x, \xi, x'+y')-a(x, \xi, x').$$

Let $a(x, \xi, x')$ be a symbol, that is, a continuous function of (x, ξ, x') in $\boldsymbol{R}^{3n}$. If m is a non-negative integer, and $0<\theta<1$, we define

$$\|a\|_m=\sup_{x,\xi,x',|\alpha|\leqq m}|\partial_\xi^\alpha a(x, \xi, x')|\langle\xi\rangle^{|\alpha|},$$
$$|a|_{m+\theta}=\sup_{x,\xi,x',|\eta|\leqq\langle\xi\rangle/2,|\alpha|=m}|\Delta_\eta\partial_\xi^\alpha a(x, \xi, x')|\langle\xi\rangle^{m+\theta}|\eta|^{-\theta},$$
$$\|a\|_{m+\theta}=\|a\|_m+|a|_{m+\theta}.$$

If t and σ are positive numbers, we define

$$\omega_\sigma(a\,;t)=\sup_{|y|\leqq t}\|\Delta_y a(x, \xi, x')\|_\sigma,$$
$$\omega'_\sigma(a\,;t)=\sup_{|y'|\leqq t}\|\Delta_{y'} a(x, \xi, x')\|_\sigma.$$

It is easy to find that $\|a\|_\sigma\leqq c\|a\|_\tau$, $\omega_\sigma(a\,;t)\leqq c\omega_\tau(a\,;t)$, $\omega'_\sigma(a\,;t)\leqq c\omega'_\tau(a, t)$ if $\sigma<\tau$, where c is a constant independent of a and t.

2. Main results. Our main results are stated as follows:

Theorem 1. *If a symbol $a(x, \xi)$ satisfies the conditions*

(a) $\|a\|_\sigma$ *is finite, and*

(b) $\omega_\sigma(a\,;t)\in L_2^*$ $(=L_2([0, 1], t^{-1}dt))$

for some $\sigma>n/2$, then the pseudo-differential operator $a(X, D)$ is bounded in $L_2(\boldsymbol{R}^n)$.

If a symbol $a(\xi, x')$ satisfies the conditions (a) *and*

(b′) $\omega'_\sigma(a\,;t)\in L_2^*$

for some $\sigma>n/2$, then the pseudo-differential operator $a(D_X, X')$ is

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bounded in $L_2(\boldsymbol{R}^n)$.

Theorem 2. *If a symbol* $a(x,\xi)$ (*or* $a(\xi,x')$) *satisfies the conditions* (a) *and* (b) (*or* (b′)) *for some* $\sigma>n+1$, *then the operator* $a(X,D)$ (*or* $a(D_X,X')$) *is bounded in* $L_p(\boldsymbol{R}^n)$ $(1<p<\infty)$.

Theorem 3. *If a symbol* $a(x,\xi,x')$ *satisfies the conditions* (a) *and*

$$\text{(c)}\quad \begin{cases} \omega_\sigma(a\,;t)^2+\omega'_\sigma(a\,;t)\in L_1^*\ (=L_1([0,1],t^{-1}dt), \\ or\ \omega_\sigma(a\,;t)+\omega'_\sigma(a\,;t)^2\in L_1^* \end{cases}$$

for some $\sigma>n$, *then the pseudo-differential operator* $a(X,D_X,X')$ *is bounded in* $L_2(\boldsymbol{R}^n)$.

Theorem 4. *If* $a(x,\xi,x')$ *satisfies the conditions* (a) *and* (c) *for some* $\sigma>n+1$, *then* $a(X,D_X,X')$ *is bounded in* $L_p(\boldsymbol{R}^n)$ $(1<p<\infty)$.

3. Comparison with the previous investigations. Assuming (a) with $\sigma=n+2$ and the condition

$$\omega_{n+2}(a\,;t)+\omega'_{n+2}(a\,;t)\leqq ct^\delta \qquad (0<\delta\leqq 1),$$

(that is, Hölder continuous case) Muramatu (Colloquium at Tokyo Univ. of Education. See also [7].) and Nagase ([8]) proved L_p-boundedness of the operator $a(X,D_X,X')$. Mossaheb-Okada ([5]) proved L_p-boundedness of the operator $a(X,D)$ under the conditions (a) with $\sigma=n+2$ and

$$\omega_{n+2}(a\,;t)\leqq C\,(\log 2/t)^{-1},$$

while Coifman-Meyer ([4]) gave the same boundedness theorem under the conditions (a) and

$$\omega_\sigma(a\,;t)\leqq c\,(\log 2/t)^{-\delta}, \qquad \delta>1/2,$$

with $\sigma=n+[n/2]+2$.

Theorem 1 is closely related with Cordes-Kato's theorem ([1], [2]), which states that the operator $a(X,D)$ is bounded in L_2 if its symbol $a(x,\xi)$ satisfies

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{(|\beta|-|\alpha|)\rho}$$

for all $|\alpha|\leqq[n/2]+1$, $|\beta|\leqq[n/2]+2$, where $0\leqq\rho<1$.

4. An interpolation theorem and some lemmas. We shall state here an auxiliary results needed in our argument.

Theorem 5. *Let X and Y be Banach spaces, and let* $H(x,x')$ *be an* $\mathcal{L}(X,Y)$*-valued strongly measurable function of* (x,x') *in* $\boldsymbol{R}^n$, *where* $\mathcal{L}(X,Y)$ *denotes the space of all bounded linear operators from X to Y. Assume that the operator T defined by*

$$(4.1)\qquad Tu(x)=\int H(x,x')u(x')dx' \qquad for\ u\in\mathcal{S}(\boldsymbol{R}^n\,;X)$$

is bounded operator from $L_2(\boldsymbol{R}^n\,;X)$ *to* $L_2(\boldsymbol{R}^n\,;Y)$, *and*

$$(4.2)\qquad \operatorname*{ess.\,sup}_{b>0,x'\in\boldsymbol{R}^n} b\int\chi_b(x-x')\sum_{1\leqq j\leqq n}\|\partial_{x'_j}H(x,x')\|_{\mathcal{L}(X,Y)}\,dx<\infty,$$

where χ_b *is the characteristic function of the set* $\{x\,;|x_j|>b$ *for some* $1\leqq j\leqq n\}$. *Then T is a bounded operator from* $L_p(R^n\,;X)$ *to* $L_p(R^n\,;Y)$ *for* $1<p\leqq 2$.

This theorem can be proved in the same way as in [6, pp. 96–97].

Lemma 1. *Let m be an integer, $0<\theta<1$, $1\leqq p\leqq 2$, $1/p+1/p'=1$, and let $\hat{f}$ be the Fourier transform of f.*

(i) *If $\hat{f}(\xi)\in W_p^m(R_\xi^n)$, then $f(x)\langle x\rangle^m \in L_{p'}(R_x^n)$.*

(ii) *If $\hat{f}(\xi)\in B_{p,p}^{m+\theta}(R_\xi^n)$, then $f(x)\langle x\rangle^{m+\theta}\in L_{p'}(R_x^n)$, where $B_{p,p}^\sigma(R^n)$ denotes the Besov spaces.*

Making use of this lemma and Hölder's inequality, we can prove the following

Lemma 2. (i) *If a symbol $a(x,\xi)$ vanishes at $|\xi|\geqq b>0$, and satisfies the condition*

$$\sup_x \|a(x,\xi)\|_{B_{p,p}^\sigma(R_\xi^n)}<\infty \tag{4.3}$$

for some $\sigma>\max(n/2, n/p)$, then $a(X,D)$ is bounded in L_p.

(ii) *If a symbol $a(\xi,x')$ vanishes at $|\xi|\geqq b>0$, and satisfies the conditions*

$$\sup_x \|a(\xi,x')\|_{B_{p',p'}^\sigma(R_\xi^n)}<\infty \qquad (1/p+1/p'=1) \tag{4.4}$$

for some $\sigma>\max(n/2, n/p')$, then $a(D_X, X')$ is bounded in L_p.

5. Sketch of the proofs. Consider first a symbol $a(x,\xi)$ satisfying (a) and (b). Let $\sigma=m+\theta$, $0<\theta<1$. Then, with the aid of the approximation theorem of symbols (see [3]), we can write as

$$a(x,\xi)=a_0(x,\xi)+a_1(x,\xi)+a_2(x,\xi),$$

where a_0, a_1, and a_2 are symbols having the following properties: $a_0(x,\xi)$ vanishes at $|\xi|\geqq 3$, while $a_1(x,\xi)$ and $a_2(x,\xi)$ vanishes at $|\xi|\leqq 2$. a_1 satisfies the conditions

$$|\partial_x^\beta\partial_\xi^\alpha a_1(x,\xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{\delta|\beta|-|\alpha|} \tag{5.1}$$

for any β and $|\alpha|\leqq m$, and

$$|\Delta_\eta\partial_x^\beta\partial_\xi^\alpha a_1(x,\xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{\delta|\beta|-|\alpha|-\theta}|\eta|^\theta \tag{5.2}$$

for any β, $|\alpha|=m$, and $|\eta|\leqq\langle\xi\rangle/2$. a_2 satisfies the conditions

$$|\partial_\xi^\alpha a_2(x,\xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}h(\langle\xi\rangle^{-\delta}) \tag{5.3}$$

for $|\alpha|\leqq m$, and

$$|\Delta_\eta\partial_\xi^\alpha a_2(x,\xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{-|\alpha|-\theta}|\eta|^\theta h(\langle\xi\rangle^{-\delta}) \tag{5.4}$$

for $|\alpha|=m$ and $|\eta|\leqq\langle\xi\rangle/2$. Here $C_{\alpha\beta}$ is a constant independent of x and ξ, δ is a constant with $0<\delta<1$, and $h(t)$ is a non-decreasing function of t with $h\in L_2^*$.

L_2-boundedness of $a_1(X,D)$ has been known (cf. see [2]. This can be proved also by using Calderón-Vaillancourt's lemma). Combining this with Theorem 5, we get L_p-boundedness of $a_1(X,D)$. Boundedness of $a_0(X,D)$ follows from Lemma 2. To prove boundedness of $a_2(X,D)$ we need the integral representation

$$a_2(X,D)u=(2\pi)^{-n/2}\int_0^1 A(t)u\,dt/t, \tag{5.5}$$

$$A(t)u(x)=\iint K(t,x,z)t^{-n}\varphi\left(\frac{x-x'}{t}-z\right)u(x')\,dz\,dx', \tag{5.6}$$

$$(5.7) \qquad K(t, x, z) = (2\pi)^{-n/2} \int e^{iz\xi} a_2(x, \xi/t) f(|\xi|) d\xi ,$$

where f is a C^∞-function of a real variable whose support is contained in the interval $[1/2, 1]$, and φ is a rapidly decreasing C^∞-function.

The operator $a(D_X, X')$ can be discussed in the same way.

Finally consider a symbol $a(x, \xi, x')$ satisfying (a) and (c). By the approximation theorem and the expansion theorem we obtain

$$a(X, D_X, X') = a_0(X, D_X, X') + a_1(X, D) + a_2(X, D) + a_3(X, D_X, X')$$

(we consider here the case where $\omega_\sigma(a; t)^2 + \omega'_\sigma(a; t) \in L_1^*$), where $a_0(x, \xi, x')$ satisfies (a) and vanishes at $|\xi| \geqq 3$, $a_1(x, \xi)$, $a_2(x, \xi)$ and $a_3(x, \xi, x')$ vanishes at $|\xi| \leqq 2$, a_1 satisfies (5.1) and (5.2), a_2 satisfies (5.3) and (5.4), and a_3 satisfies (5.3) and (5.4) with $h \in L_1^*$. The rest of the proof is the same as that of the case $a(x, \xi)$.

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