7. On a Direct Method of Constructing Multi-Soliton Solutions

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1. In this note we describe a direct method of constructing multi-soliton solutions of the nonlinear equations solvable by the inverse scattering method. In particular we consider the Zakharov-Shabat equations, the sine-Gordon equation and the equation of motion of the Toda Lattice.

2. Construction of simultaneous solutions. The above equations are expressed as the commutativity of two linear operators.

i) The Zakharov-Shabat equations [8]: they are a class of nonlinear differential equations for functions $u_j(x,y,t)$, $0\leq j\leq n-2$, $v_k(x,y,t)$, $0\leq k\leq m-2$ expressed as

$$[L-\partial/\partial y, M-\partial/\partial t]=0$$

where $L=\sum_{j=0}^{n} u_j D^j$, $M=\sum_{j=0}^{m} v_j D^j$, $D=\partial/\partial x$ and $u_n, u_{n-1}, v_m, v_{m-1}$ are constants. This class includes the Korteweg-de Vries equation, the Boussinesq equation, the two-dimensional Korteweg-de Vries equation as special cases.

ii) The sine-Gordon equation [1], [7]

$$u_{\xi\eta}+\sin u=0:$$

this equation is expressed as

$$[L,M]=0$$

where

$$L=\partial/\partial\xi-2^{-1}i\lambda\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-2^{-1}iu_\xi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$M=\partial/\partial\eta-2^{-1}i\lambda^{-1}\begin{pmatrix}0 & \exp(iu)\\ \exp(-iu) & 0\end{pmatrix}.$$

iii) The equation of motion of the Toda Lattice [6], [2], [4]

$$\partial Q_n/\partial t=P_n,$$
$$\partial P_n/\partial t=\exp(Q_{n-1}-Q_n)-\exp(Q_n-Q_{n+1}),\qquad n\in Z,$$

or

$$\partial a_n/\partial t=2a_n(b_{n+1}-b_n),$$
$$\partial b_n/\partial t=2a_n(a_n-a_{n-1})$$

where

$$a_n=4^{-1}\exp\{(Q_{n-1}-Q_n)/4\},\qquad b_n=-2^{-1}P_n:$$

this equation is expressed as

$$[L, M-\partial/\partial t]=0$$

by linear operators

$$(Lu)(n)=u(n+1)+b_n u(n)+a_{n-1}u(n-1),$$
$$(Mu)(n)=u(n+1)+b_n u(n)-a_{n-1}u(n-1)$$

acting on the space of sequences $u=\{u(n)\}$.

Let N be any positive integer, $\alpha_1, \cdots, \alpha_N, \beta_1, \cdots, \beta_N$ be mutually distinct complex numbers and $c_1, \cdots, c_N$ be any complex numbers.

We consider the functions of the following forms

(a.1) $\Phi(x, y, t, \lambda)=(\lambda^N+\sum_{j=0}^{N-1}\phi_j(x, y, t)\lambda^j)\exp(\lambda x+P(\lambda)y+Q(\lambda)t)$ where $\lambda\in\boldsymbol{C}$ and $P(\lambda)=\sum_{j=0}^{n}p_j\lambda^j$, $Q(\lambda)=\sum_{j=0}^{m}q_j\lambda^j$ are any polynomials with constant coefficients,

(a.2) $\Phi_n(\xi, \eta, \lambda)=(\lambda^N+\sum_{j=0}^{N-1}\phi_{nj}(\xi, \eta)\lambda^j)\exp(\lambda\xi+\lambda^{-1}\eta)$, $n=1, 2$,

(a.3) $\Phi_n(t, \lambda)=\lambda^n(\lambda^N+\sum_{j=0}^{N-1}\phi_{nj}(t)\lambda^j)\exp(t(\lambda-\lambda^{-1}))$, $n\in\boldsymbol{Z}$.

For these functions we impose the following conditions

(b.1) $\Phi(x, y, t, \alpha_j)=c_j\Phi(x, y, t, \beta_j)$, $1\le j\le N$,

(b.2) $\Phi_n(\xi, \eta, \alpha_j)=(-1)^{n-1}c_j\Phi_n(\xi, \eta, \beta_j)$, with $\beta_j=-\alpha_j$, $1\le j\le N$, $n=1, 2$,

(b.3) $\Phi_n(t, \alpha_j)=c_j\Phi_n(t, \beta_j)$, with $\beta_j=\alpha_j^{-1}$, $1\le j\le N$, $n\in\boldsymbol{Z}$.

Proposition 1. *By requirements* (b.1) (*resp.* (b.2), (b.3)), *the function* $\Phi(x, y, t, \lambda)$ (*resp.* $\Phi_n(\xi, \eta, \lambda)$, $\Phi_n(t, \lambda)$) *is uniquely determined.*

For the Toda Lattice, for example, relations (b.3) are equivalent to the following system of linear equations for unknowns $\phi_{nj}(t)$, $0\le j\le N-1$:

$$\sum_{k=0}^{N-1}(\alpha_j^{n+k}e(\alpha_j)-c_j\alpha_j^{-(n+k)}e(-\alpha_j))\phi_{nk}$$
$$=-\alpha_j^{n+N}e(\alpha_j)+c_j\alpha_j^{-(n+N)}e(-\alpha_j), \qquad j=1, \cdots, N,$$

where $e(\lambda)=\exp(t(\lambda-\lambda^{-1}))$. We can show that the coefficient matrix of this system is nonsingular, hence functions $\phi_{nj}(t)$ are uniquely determined.

By similar arguments we can show

Proposition 2. (c.1) *The function* $\Phi(x, y, t, \lambda)$ *satisfies the equations*

$$\sum_{j=0}^{n}u_j(x, y, t)D^j\Phi=\partial\Phi/\partial y,$$
$$\sum_{j=0}^{m}v_j(x, y, t)D^j\Phi=\partial\Phi/\partial t$$

with coefficients u_j, v_j *uniquely determined from* ϕ_j, p_j, q_j, *in particular* $u_n=p_n$, $u_{n-1}=p_{n-1}$, $v_m=q_m$, $v_{m-1}=q_{m-1}$.

(c.2) *The vector-valued function* $\Phi(\xi, \eta, \lambda)={}^t(\Phi_1(\xi, \eta, \lambda), \Phi_2(\xi, \eta, \lambda))$ *satisfies the equations*

$$L\Phi=0, \qquad M\Phi=0$$

with coefficient $u=-i\log(\phi_{1,0}/\phi_{2,0})$.

(c.3) *The sequence of functions* $\Phi(t, \lambda)=\{\Phi_n(t, \lambda)\}$ *satisfies*

$$L\Phi=(\lambda+\lambda^{-1})\Phi,$$
$$M\Phi=\partial\Phi/\partial t$$

with coefficients $a_n=\phi_{n+1,0}/\phi_{n,0}$, $b_n=\phi_{n,N-1}-\phi_{n+1,N-1}$.

Thus we have

Theorem. *The functions* $u_j(x, y, t)$, $v_j(x, y, t)$ (*resp.* $u(\xi, \eta)$, $a_n(t)$, $b_n(t)$) *constructed above are the solution of the equation* i) (*resp.* ii), iii)).

Remark 1. In i), replacing scalar-valued functions u_j, v_k by matrix-valued functions, we obtain matrix-Zakharov-Shabat equations, in which we regard v_n, v_m as non-singular constant diagonal matrices and u_{n-1}, v_{m-1} as functions of (x, y, t). The construction of $\Phi(x, y, t, \lambda)$ is similar to the scalar case. Details will appear elsewhere.

Remark 2. Our construction for the Zakharov-Shabat equations corresponds to the algebro-geometric construction of Kricheber [3] and Manin [5]. However they do not give explicit expressions of the solution.

3. Expressions of solutions. We can show that our solutions are the same with multi-soliton solutions obtained by the inverse scattering method.

We show this for the Toda Lattice. By Cramér's formula, functions $\phi_{n,0}$ and $\phi_{n,N-1}$ are expressed as

$$\phi_{n,0}=(-1)^N \det(f^{(n+1)}, \cdots, f^{(n+N)})/\det(f^{(n)}, \cdots, f^{(n+N-1)}),$$

$$\phi_{n,N-1}=-\det(f^{(n)}, \cdots, f^{(n+N-2)}, f^{(n+N)})/\det(f^{(n)}, \cdots, f^{(n+N-1)})$$

where

$$f^{(k)}={}^t(\alpha_1^k e(\alpha_1)-c_1\alpha_1^{-k}e(-\alpha_1), \cdots, \alpha_N^k e(\alpha_N)-c_N\alpha_N^{-k}e(-\alpha_N)).$$

By direct calculations, we have

$$\det(f^{(n)}, \cdots, f^{(n+N-1)})$$
$$=\exp\left(\sum_{l=1}^N t(\alpha_l-\alpha_l^{-1})\right)\prod_{a>b}(\alpha_a-\alpha_b)$$
$$\times\det\left(\delta_{jk}-\frac{c_j\alpha_j^{-2n}}{\alpha_j^{-1}-\alpha_k}\frac{g(\alpha_j^{-1})}{\dot{g}(\alpha_k)}\exp(-t(\alpha_j+\alpha_k-\alpha_j^{-1}-\alpha_k^{-1}))\right)$$

where $g(\lambda)=\prod_{j=1}^N(\lambda-\alpha_j)$, $\dot{g}=dg/d\lambda$.

Using this expression, we can show that the solution of iii) which we have constructed is the same with that constructed by the inverse scattering method [2], [4].

Details and further extensions will appear elsewhere.

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