62. On the Number of Conjugate Classes of Maximal Subgroups in Finite Groups

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1. Introduction. M. Numata [1] proved that the nilpotent length of a finite solvable group is at most one plus the number of conjugate classes of the non-normal maximal subgroups.

In this paper we shall prove the following two theorems. One of them partially extends Numata's result.

Theorem 1. *Suppose every non-normal maximal subgroup of a finite group G has the same order. Then G is solvable and the nilpotent length of G is at most two.*

Theorem 2. *The number of conjugate classes of maximal subgroups of a finite non-abelian simple group is at least three.*

Alternating group A_5 has just three conjugate classes of maximal subgroups of it. So the number three in Theorem 2 is best possible. An example related to Theorem 2 is found in the paper [2] due to Goldschmidt, which gives a group-theoretic proof of Burnside's theorem concerning the solvability of groups of order $p^a q^b$ for odd primes p, q. In the paper it is shown that if G is a minimal counter example, then G is simple and the number of conjugate classes of maximal subgroups of G is two. Hence the proof may also be completed by Theorem 2.

2. Proof of the theorems. Let G be a permutation group on Ω, denoted by G^{Ω}, and H be a subgroup of G. We denote by $I(H)$ a set of the points of Ω left fixed by H. We need the following well-known lemma, which is proved by using Witt's lemma [3, P 20], and Lemma 6 of [4].

Lemma. *Let G be a transitive permutation group on Ω and p be a prime. Suppose P is a p-subgroup of G of maximal order which fixes at least two points. Then $N_G(P)$ is transitive on $I(P)$.*

Proof of Theorem 1. We may suppose that there exists a non-normal maximal subgroup H in G. Let p be a prime dividing $|G:H|$ and let P be a Sylow p-subgroup of G. If $G \gneqq N_G(P)$, then there exists a maximal subgroup L such the $L \geqq N_G(P)$. Since $L \geqq N_G(P)$, we obtain $L = N_G(L)$ and so L is a non-normal maximal subgroup of G. Hence $|L| = |H|$, contrary to our choice of p. Consequently $G \triangleright P$. Let $\bar{L} = L/P$ be any maximal subgroup of $\bar{G} = G/P$. Since p does not divide $|G:L|$,

L is a normal subgroup of G and $\bar{G} \triangleright \bar{L}$. Hence $\bar{G}$ is nilpotent and the theorem is proved.

Proof of Theorem 2. Assume the theorem is false for a simple group G. By Theorem 1, G possesses two conjugate classes of maximal subgroups. We denote them by $\Gamma = \{L^x \mid x \in G\}$ and $\Omega = \{M^x \mid x \in G\} = \{M = M_1, M_2, \cdots, M_m\}$. Let $|\Gamma| = |G : L| = l$, then it is immediate that l and m are relatively prime. We assume that $l > m$.

Since G^Ω is not a Frobenius group, we may assume that $M_1 \cap M_2$, which is the stabilizer of M_1 and M_2, is not trivial. Let p be any prime dividing $|M_1 \cap M_2|$ and let P be a p-subgroup of G^Ω of maximal order fixing at least two points. We may assume that $P \leqq M_1$. By lemma $N_G(P)$ is transitive on $I(P)$ and so $N_G(P)$ fixes no points of Ω. Thus $N_G(P)$ is not contained in M^x for every $x \in G$, and then $N_G(P) \leqq L^x$ for some $x \in G$. Then it follows that $L^x \cap M_1 \geqq P$ and so $|L^x \cap M_1| = |L \cap M_1|$ is divided by $|P|$. On the other hand the order of a Sylow p-subgroup of $M_1 \cap M_2$ is not greater than $|P|$. Thus $|L \cap M_1|$ is divided $|M_1 \cap M_2|$. Since l and m are relatively prime, $G = LM_1$ and $|M_1 : L \cap M_1| = l$. Now $|M_1| = l|L : M_1| = |M_1 : M_1 \cap M_2||M_1 \cap M_2|$ and $|M_1 \cap M_2|$ divides $|L : M_1|$. Therefore l divides $|M_1 : M_1 \cap M_2|$. Since $l > m$, we have $|M_1 : M_1 \cap M_2| > m$. This implies that the length of the orbit of M_1^Ω containing M_2 is at least $m+1$, contrary to $m = |\Omega|$.

References

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