51. *Construction of Complex Structures on Open Manifolds**⁾

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1. Introduction. In 1951, in his book [6] N. Steenrod conjectured, "it seems highly unlikely that every almost complex manifold has a complex analytic structure".

In [7] Van de Ven showed the existence of a compact almost complex manifold of dimension 4 which does not admit any complex structure.

Recently S.-T. Yau [8] and N. Brotherton [2] have shown some examples of compact parallelizable manifolds of dimension 4 which do not admit any complex structure.

On the other hand, in [3] M. Gromov has shown a method to obtain complex structures on a special almost complex manifold. As a corollary, he has shown that on an open manifold of dimension 4, any almost complex structure is homotopic to a complex one.

In this note we shall improve a little on Gromov's result on the construction of complex structures on open manifolds. As a corollary we shall prove that on an open 6-dimensional manifold, any almost complex structure is homotopic to a complex one.

We study this problem within the frame work of A. Haefliger [4], [5] which permits one to view the problem as a lifting problem in homotopy theory.

The interest of Dr. K. Nakajima in the integrability of almost complex structures stimulated the appearance of the present note.

2. Preliminaries. We now give a brief recall on Haefliger's work [4], [5] that are needed here. Let Γ_q^C denote the topological groupoid of germs of local complex analytic automorphisms of C^q, and let $B\Gamma_q^C$ denote a classifying space for Γ_q^C-structures. The differential induces a continuous homomorphisms

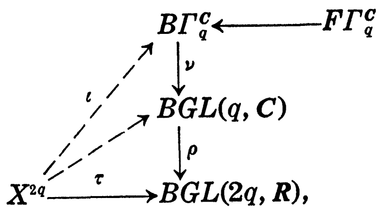
$$\nu : \Gamma_q^C \to GL(q, C),$$

hence also a continuous map

$$\nu : B\Gamma_q^C \to BGL(q, C).$$

We convert this map to a fibration and write $F\Gamma_q^C$ for the homotopy fibre. Consider the following diagram:

*⁾ Dedicated to Professor A. Komatu for his 70th birthday.



where X^{2q} is an open $2q$-manifold and τ classifies the tangent bundle of X^{2q}. It is a standard bundle theory that homotopy classes of liftings of τ to $BGL(q, C)$ correspond to homotopy classes of almost complex structures on X^{2q}. As a special case of the results of Haefliger [4], [5], homotopy classes of liftings ι of τ to $B\Gamma_q^C$ are in one-to-one correspondence with integrable homotopy classes of complex analytic structures on X^{2q}.

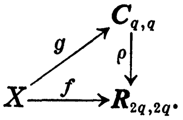
Thus for the integrability problem of almost complex structures, it is important to understand the homotopy fibre $F\Gamma_q^C$. We have the following

Connectivity Theorem ([1]). *$F\Gamma_q^C$ is q-connected.*

3. Construction of complex structures. We shall call a continuous mapping of a topological space X into a polyhedron $|K|$ *k-compressible,* if it is homotopic to a map with image $f(X)$ in the k-dimensional skeleton of $|K|$.

Let X be an open manifold of dimension $m=2q$, equipped with an almost complex structure σ.

Let $\boldsymbol{R}_{2q,2q}$ be the Grassmannian manifold of all $2q$-dimensional vector subspaces in $\boldsymbol{R}^{4q}$ and $\boldsymbol{C}_{q,q}$ be the complex Grassmannian manifold of all q-dimensional vector subspaces in $\boldsymbol{C}^{2q}$. Then we have the canonical map $\rho: \boldsymbol{C}_{q,q} \to \boldsymbol{R}_{2q,2q}$. In this case, the tangent bundle $T(X)$ is induces by a classifying map $f: X \to \boldsymbol{R}_{2q,2q}$ and this map f can be lifted to $\boldsymbol{C}_{q,q}$ as follows:



Theorem. *If the map g is $(q+1)$-compressible, then on X there exists a complex structure which is homotopic to the given almost complex structure σ corresponding to g.*

Proof. By the assumption the image $g(X)$ of g is contained in the $(q+1)$-skeleton of $C_{q,q}$. The obstruction of lifting of g to $B\Gamma_q^C$ is in $H^i(X, \pi_{i-1}(F\Gamma_q^C))$, $i=1, 2, \cdots$. However, by Connectivity Theorem, we have $\pi_{i-1}(F\Gamma_q^C)=0$ for $i \leqq q+1$. Therefore,

$$H^i(X, \pi_{i-1}(F\Gamma_q^C))=0, \qquad 1 \leqq i \leqq q+1.$$

Since $g(X) \subset (q+1)$-skeleton of $\boldsymbol{C}_{q,q}$, further obstruction is zero. Thus we have obtained the theorem.

Corollary. a) *Let X be an open $2q$-dimensional manifold, and X be homotopically equivalent to a polyhedron $|K|$ of dimension $\leqq 2s+1 \leqq q+4$. Then for any almost complex structure σ on X, for which the ring generated by the Chern classes has no nontrivial elements in $H^{2s}(X, Z)$, one can find a complex structure homotopic to σ.*

b) *In particular, on an open 6-dimensional manifold, any almost complex structure is homotopic to a complex one.*

Proof. a) Let $g: X \to C_{q,q}$ be a map corresponding to the almost complex structure σ. It suffices to show that g is $(q+1)$-compressible. By the assumption g is $(2s-1)$-compressible. Therefore, g is $(2s-2)$-compressible. However, $2s-2 \leqq q+1$. Thus we have obtained a).

b) is easily obtained.

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