48. On the Asymptotic Behavior of Iterates of Nonexpansive Mappings in Uniformly Convex Banach Spaces

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§1. **Introduction.** Let X^* be the dual of a Banach space X, and denote the value of $x^* \in X^*$ at $x \in X$ by (x, x^*). The *duality mapping* F from X into X^* is defined by $F(x)=\{x^* \in X^* : (x, x^*) =\|x\|^2=\|x^*\|^2\}$ for $x \in X$. The norm of X is said to be *Fréchet differentiable* if for each x in the unit sphere U of X $\lim_{t\to 0} t^{-1}(\|x+ty\| -\|x\|)$ exists uniformly in $y \in U$. It is known (e.g. see [3]) that the norm of X is Fréchet differentiable if and only if F is single-valued and norm to norm continuous from X to X^*. Let C be a subset of X. A mapping $T: C \to C$ is said to be *nonexpansive* on C, or $T \in \mathrm{Cont}\,(C)$ if $\|Tx-Ty\| \leqq \|x-y\|$ for all $x, y \in C$. We denote by F_T the set of all fixed points of T, and by $\omega_w(x)$ the set of all weak subsequential limits of $\{T^n x\}$. We set $N=\{1, 2, \cdots\}$ and $N_0=N \cup \{0\}$.

Throughout this note let X denote a uniformly convex real Banach space, C a nonempty closed convex subset of X, and let $x \in C$ and $T \in \mathrm{Cont}\,(C)$. Our main result of this note is the following; and the sketch of the proof is given in §2. The complete proof of our result will be given in the subsequent paper.

Theorem. *If the norm of X is Fréchet differentiable, then the following four conditions are equivalent:*

(a) *$T^n x$ converges weakly as $n \to \infty$,*

(b) *$T^n x$ converges weakly as $n \to \infty$ to a fixed point of T,*

(c) *$F_T \neq \emptyset$ and $\omega_w(x) \subset F_T$,*

(d) *$F_T \neq \emptyset$ and $w\text{-}\lim_{n\to\infty} (T^n x - T^{n+1} x)=0$.*

Remark. This theorem contains the case that X and X^* are uniformly convex. If X is a Hilbert space, this theorem is due to Pazy [7], Bruck [2] and Schöneberg [8]. If $X=L^p(\Omega)$, $1<p<\infty$, then the theorem was announced by Baillon [1]. On the other hand, Miyadera [5], [6] has recently given another extension of a result of [2] and [7], that is our result holds under the condition that X is uniformly convex and F is weakly continuous at 0.

§2. **Sketch of proof.** Our theorem will follow from the following two propositions.

Proposition 1. *If the norm of X is Fréchet differentiable, we have that $(u-v, F(f-g))=0$ for all $u, v \in \mathrm{clco}\,\omega_w(x)$ and $f, g \in F_T$,* *where* $\operatorname{clco} \omega_w(x)$ *denotes the closed convex hull of* $\omega_w(x)$.

Proposition 2. *Let* $\{n_k\}$ *be a sequence in* N *such that* $n_k \to \infty$ *and* $w\text{-}\lim_{k\to\infty} T^{n_k}x = y$. *If* $\{T^n x\}$ *is bounded and* $w\text{-}\lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$, *then* $y \in F_T$.

Now, to show the propositions we need the next lemma which follows from the uniform convexity of X.

Lemma 1. *Let* $p>1$. *Let* u_n^α *and* v_n^α *be elements of* X *defined for* $n \in N$ *and* $\alpha \in A$, *where* A *is a nonempty set. Put* $a_k^\alpha = 2^{-1}(\|u_n^\alpha\|^p + \|v_n^\alpha\|^p) - \|2^{-1}(u_n^\alpha + v_n^\alpha)\|^p$. *Suppose that* $\{u_n^\alpha\}$ *and* $\{v_n^\alpha\}$ *are bounded.*

(i) *If* $\lim_{n\to\infty} a_n^\alpha = 0$ *uniformly in* α, *then* $\lim_{n\to\infty}\|u_n^\alpha - v_n^\alpha\| = 0$ *uniformly in* α.

(ii) *If* $\lim_{n\to\infty} n^{-1}\sum_{k=1}^n a_k^\alpha = 0$ *uniformly in* α, *then* $\lim_{n\to\infty} n^{-1} \times \sum_{k=1}^n \|u_k^\alpha - v_k^\alpha\|^p = 0$ *uniformly in* α.

Lemma 2. *Let* $f \in F_T$. *Then for each* $k \in N$, $\lim_{n\to\infty}\|T^m z_k(n) - z_k(n+m)\| = 0$ *uniformly in* $m \in N$, *where* $z_k(n) = 2^{-k}(T^n x - f) + f$.

Sketch of Proof. Set $u_{n,k}^m = T^m z_k(n) - z_{k-1}(n+m)$ and $v_{n,k}^m = -T^m z_k(n) + f$. From the nonexpansiveness of T it follows that $\{u_{n,k}^m\}$ and $\{v_{n,k}^m\}$ are bounded. Since $u_{n,k}^m - v_{n,k}^m = 2(T^m z_k(n) - z_k(n+m))$, to prove the lemma we may show that for each k $\lim_{n\to\infty}\|u_{n,k}^m - v_{n,k}^m\| = 0$ uniformly in m. To this end, by virtue of Lemma 1 (i) it suffices to show that for each k

$$\lim_{n\to\infty} 2^{-1}(\|u_{n,k}^m\|^p + \|v_{n,k}^m\|^p) - \|2^{-1}(u_{n,k}^m + v_{n,k}^m)\|^p = 0$$

uniformly in m. But this will be proved by induction on k.

Lemma 3. *Suppose that the norm of* X *is Fréchet differentiable. Then* $\lim_{n\to\infty}(T^n x - f, F(f-g))$ *exists for every* $f, g \in F_T$.

Proof. Let $f, g \in F_T$. Set $b_{k,n} = 2^k(\|z_k(n) - g\| - \|f-g\|)$. Using Lemma 2, we can obtain that $\lim_{n\to\infty} b_{k,n}$ exists for each k. Moreover $\lim_{k\to\infty} b_{k,n}$ exists uniformly in n. Indeed, since $\|T^n x - f\| \leqq \|x - f\|$ for all n, the Fréchet differentiability of the norm of X implies that $\lim_{k\to\infty} b_{k,n} = \lim_{k\to\infty} 2^k(\|f - g + 2^{-k}(T^n x - f)\| - \|f-g\|)$ exists uniformly in n. Therefore, $\lim_{n\to\infty}\lim_{k\to\infty} b_{k,n}$ exists, and hence

$$\lim_{n\to\infty}(T^n x - f, F(f-g)) = \|f-g\| \lim_{n\to\infty}\lim_{k\to\infty} b_{k,n}$$

exists.

Proof of Proposition 1. Let $f, g \in F_T$. If $u, v \in \omega_w(x)$, there exist subsequences $\{n'\}$ and $\{n''\}$ of $\{n\}$ such that $T^{n'}x \xrightarrow[w]{} u$ and $T^{n''}x \xrightarrow[w]{} v$. By Lemma 3 we have

$$\begin{aligned}(u-f, F(f-g)) &= \lim_{n'\to\infty}(T^{n'}x - f, F(f-g)) \\ &= \lim_{n''\to\infty}(T^{n''}x - f, F(f-g)) \\ &= (v-f, F(f-g)),\end{aligned}$$

and hence $(u-v, F(f-g)) = 0$ for $u, v \in \omega_w(x)$. But this is also true for $u, v \in \operatorname{clco}\omega_w(x)$, for the function $p(u) = (u - u_0, v^*)$ is continuous and affine on X for each $u_0 \in X$ and $v^* \in X^*$.

Next, to establish Proposition 2, we start with the following notation: Let $k\in N$. For $n\in N_0$ and $\alpha=(n_1, n_2, \cdots, n_k)\in N_0^k$ define

$$v(n, \alpha)=k^{-1}\sum_{i=1}^{k} T^{s_i+n}x,$$

where $s_i=n_1+n_2+\cdots+n_i$, $i=1, 2, \cdots, k$.

Lemma 4. *Let $p>1$. Suppose that $\{T^n x\}$ is bounded: $\|T^n x\|\leqq M$. Then for each $q\in N$ $\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1}\|Tv(i, \alpha)-v(i+1, \alpha)\|^p=0$ uniformly in $\alpha\in N_0^k$, where $k=2^q$.*

Sketch of Proof. Set $k=2^q$ and $j=2^{q-1}$. For $n\in N_0$ and $\alpha=(n_1, n_2, \cdots, n_k)\in N_0^k$ define $u_n=Tv(n, \alpha)-v(n+1, \alpha')$ and $v_n=-Tv(n, \alpha)+v(n+1, \alpha'')$, where $\alpha'=(n_1, n_2, \cdots, n_j)$ and $\alpha''=(n_1+\cdots+n_{j+1}, n_{j+2}, \cdots, n_k)$. $\{u_n\}$ and $\{v_n\}$ are bounded, for $\{T^n x\}$ is bounded. Note that $u_n-v_n=2(Tv(n, \alpha)-v(n+1, \alpha))$, since $v(n, \alpha)=2^{-1}(v(n, \alpha')+v(n, \alpha''))$. We can then show that for all n and all $\lambda, \mu>0$ with $\lambda+\mu=1$

(1) $$n^{-1}\sum_{i=0}^{n-1} a_i^\alpha \leqq (n^{-1}+(\lambda^{1-p}-1))M^p+2^{-1}\mu^{1-p}n^{-1}\sum_{i=0}^{n-1} Q_i(\alpha).$$

Here a_i^α is defined in Lemma 1 and $Q_i(\alpha)=\|Tv(i, \alpha')-v(i+1, \alpha')\|^p+\|Tv(i, \alpha'')-v(i+1, \alpha'')\|^p$. If $q=1$, $Q_i(\alpha)=0$ because $\alpha'=n_1$ and $\alpha''=n_1+n_2$, and hence the left hand of (1) vanishes as $n\to\infty$ uniformly in $\alpha\in N_0^2$. By Lemma 1 (ii) we obtain that $\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1}\|u_i^\alpha-v_i^\alpha\|^p=0$ uniformly in $\alpha\in N_0^2$. This proves that the lemma is true for $q=1$. Next, assume that the lemma is true for $q-1$. Then, since $\alpha', \alpha''\in N_0^j$ for $\alpha\in N_0^k$, the second term on the right of (1) vanishes as $n\to\infty$ uniformly in $\alpha\in N_0^k$ by the inductive hypothesis, and so the left hand of (1) vanishes as $n\to\infty$ uniformly in α. Hence by Lemma 1 (ii) again we see that our assertion is true for q. Thus the lemma will be proved by induction on q.

Let $X_k=X$ for $k\in N$ and let $Y_n=\prod_{k=1}^{n} X_k$ and $Y_\infty=\prod_{k=1}^{\infty} X_k$. Y_n is a Banach space with the norm $|||u|||_n=\max_{1\leqq i\leqq n}\|u^i\|$, $u=(u^1, u^2, \cdots, u^n)$ for each $n\in N$. For $u=(u^1, u^2, \cdots, u^n, \cdots)\in Y_\infty$ we set $|||u|||_\infty=\sup_{i\geqq 1}\|u^i\|$ and $u|_{Y_n}=(u^1, u^2, \cdots, u^n)$. The next lemma is a slight generalization of a result of Kakutani [4]; and its proof will be done with a little change as in [3].

Lemma 5. *Let $u\in Y_\infty$. Then for each n and each sequence $\{u_m\}$ in Y_∞ with $u_m|_{Y_n}\xrightarrow[w]{} u|_{Y_n}$ in Y_n and $\sup_m\{|||u_m|||_\infty, |||u|||_\infty\}=M<\infty$, we can extract a subsequence $\{m_k\}$ of $\{m\}$ such that for all $k, j\geqq 1$*

$$|||k^{-1}(u_{m_j}+u_{m_{j+1}}+\cdots+u_{m_{j+k-1}})|_{Y_n}-u|_{Y_n}|||_n\leqq K(k)$$

where $K(k)$ is a constant independent of n and j such that $K(k)\to 0$ as $k\to\infty$.

Proof of Proposition 2. Set $z_k^i=T^{n_k+i}x$. Consider the sequence $\{u_k\}$ in Y_∞ with $u_k=(z_k^1, z_k^2, \cdots, z_k^n, \cdots)$. Obviously, $|||u_k|||_\infty\leqq M$ since $\|T^n x\|\leqq M$. From the assumption that $T^{n_k}x\xrightarrow[w]{} y$ and $T^n x-T^{n+1}x\xrightarrow[w]{} 0$ it follows that $u_k|_{Y_n}\xrightarrow[w]{} u|_{Y_n}$ in Y_n for each $n\in N$, where $u=(y, y, \cdots)$

$\in Y_\infty$. Hence by Lemma 5 there is a subsequence $\{m_k\}$ of $\{n_k\}$ such that for $i=0,1,\cdots,n$

$$\|k^{-1}(T^{m_1+i}x+\cdots+T^{m_k+i}x)-y\|\leqq K(k),$$

or equivalently $\|v(i,\alpha_k)-y\|\leqq K(k)$, where $\alpha_k=(m_1, m_2-m_1,\cdots,m_k-m_{k-1})$. Hence we have that $\|Ty-y\|\leqq 2K(k)+\|Tv(i,\alpha_k)-v(i+1,\alpha_k)\|$ for $i=0,1,\cdots,n-1$. Summing with respect to i and dividing by n, we obtain that

$$\|Ty-y\|\leqq 2K(k)+n^{-1}\sum_{i=0}^{n-1}\|Tv(i,\alpha_k)-v(i+1,\alpha_k)\|.$$

Using Hölder's inequality, we see from Lemma 4 that the second term on the right of the above inequality vanishes as $n\to\infty$ if $k=2^q$. Moreover, since $K(k)\to 0$ as $k\to\infty$, it follows that $Ty=y$, and hence $y\in F_T$.

Proof of Theorem. Note that $\{T^n x\}$ is bounded if and only if $F_T\neq\emptyset$. Obviously, (b) implies (a) and (a) implies (d). It is a direct consequence of Proposition 2 that (d) implies (c). Finally, to prove that (c) implies (b) we may show that $\omega_w(x)$ is a singleton. To this end let $u, v\in\omega_w(x)$. Since $\omega_w(x)\subset F_T$ by hypothesis, we have $u, v\in F_T$, and hence $\|u-v\|^2=(u-v, F(u-v))=0$ by Proposition 1. This gives that $u=v$, and so $\omega_w(x)$ is a singleton.

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