46. *Global Branching Theorem for Spatial Patterns of Reaction-Diffusion System*

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Introduction. In this note, we show the main idea to prove the existence of the global branch for spatial patterns of reaction-diffusion system as follows

$$(1.1)\qquad 0=\beta^2 u_{xx}+f(u,v),$$

$$x\in I \parallel (0,1)$$

$$(1.2)\qquad 0=\frac{1}{\alpha}v_{xx}+g(u,v)$$

$$(1.3)\qquad u_x(0)=u_x(1)=v_x(0)=v_x(1)=0,$$

where $1/\alpha$ and β^2 are diffusion coefficients. Our principal assumptions are

(A-1) (1) has a invariant rectangle R (see [1]).

(A-2) (1) has a unique positive constant solution $(\bar{u},\bar{v})\in R$ such that Jacobi matrix of F at $(\bar{u},\bar{v})$ is stable, i.e., real parts of its eigenvalues are negative, where $F={}^t(f(u,v),g(u,v))$.

(A-3) 0-level curve of f is sigmoidal and that of g intersects it transversally (see [2, Figs. 1–3]).

(A-4) α is fixed to be sufficiently small.

This type of system satisfying (A-1)–(A-4) appears in many fields such as population dynamics, morphogenesis and so on (see [3], [4]). Under these assumptions, we can obtain the global result Theorem 3 for the bifurcating branch from $(\bar{u},\bar{v})$ when β varies as a bifurcation parameter. We note that Theorem 3 says not only the global existence of the bifurcating branch but also its asymptotic behavior when β tends to zero.

Remark 1. We use the word "exist globally" in the sense that the bifurcating branch in $\boldsymbol{R}^+\times(H^2_N(I))^2$*) can be extendible for any small β, i.e., the β-section of it for any small β is not empty and contains non-trivial solutions.

For the local bifurcation problem of (1), many works have been done and for the details, see, for instance, [5]. When β leaves the critical point, we have, in general, little informations about how the bifurcating branch changes, however, if the above four assumptions are satisfied, we can describe the global picture of the bifurcating

*) $\boldsymbol{R}^+=(0,+\infty)$. $H^2_N(I)$=closure of $\{\cos(n\pi x)\}_{n=0}^{\infty}$ in the usual Sobolev space $H^2(I)$.

branch rather completely through the *shadow system* (3) in the following. Replacing u and v by $u+\bar{u}$ and $v+\bar{v}$ for convenience (we use the same notation (u, v) for new dependent variables), we can see that (1) becomes

$$0=\beta^2 u_{xx}+\hat{f}(u, v) \tag{2.1}$$

$$0=\frac{1}{\alpha}v_{xx}+\hat{g}(u, v), \tag{2.2}$$

where $\hat{f}(u, v)=f(u+\bar{u}, v+\bar{v})$ and $\hat{g}(u, v)=g(u+\bar{u}, v+\bar{v})$. When α is sufficiently small, v must be nearly flat because of zero flux boundary conditions and (A-1), so the limit of (2) (as $\alpha \downarrow 0$) becomes the following system for the unknowns (u, ξ)

$$0=\beta^2 u_{xx}+\hat{f}(u, \xi) \tag{3.1}$$

$$0=\int_0^1 \hat{g}(u, \xi)dx \tag{3.2}$$

subject to zero flux boundary conditions, where $v=\xi$ is a constant function. We call (3) the *shadow system* of (2).

The asymptotic analysis with respect to α (for fixed β) was done by Keener [6], but it didn't treat the global branching problem. Singular perturbation analysis with respect to β has been done by Mimura-Hosono-Tabata [7] using Fife's method [8], which plays an important role in the following discussions. Some parts of the following results were stated in [2] and [9], and the extended paper including proofs will be reported elsewhere.

Here we introduce some notations: $\hat{C}_\alpha^n$ (or $\hat{C}_0^n$) denotes the closure of non-trivial solutions of (2) (or (3)) in $\boldsymbol{R}^+\times(H_N^2(I))^2$, respectively, which contains n-th bifurcation point from the trivial branch. (See [2], [9].)

§1. Existence of the global branch for the shadow system. First we study the global properties of the shadow system (3), because it inherits the essential features of (2) for small α, especially for β bounded away from zero, we get the following approximation lemma.

Lemma 1. *For any positive β_1 and ε, there exists some positive constant α_1 such that $\hat{C}_\alpha^n$ belongs to the ε-nbd. of $\hat{C}_0^n$ when both branches are restricted to the space $[\beta_1, \infty)\times(H_N^2(I))^2$ for any α with $0<\alpha\leqq\alpha_1$.*

Under some technical assumptions, the problem of finding the form of $\hat{C}_0^n$ is reduced to study the intersection of two surfaces in $\boldsymbol{R}^3$, which strongly depend on the nonlinearity F (see [9]). From (A-3) and homotopy invariance of the degree, we can get

Theorem 1. *Under the assumptions* (A-1)–(A-4), *$\hat{C}_0^n$ exists globally with respect to β.*

Remark 2. The result of Theorem 1 is sufficient to guarantee the global existence of the branch $\hat{C}_\alpha^n$.

§2. Singular perturbation solution and its local uniqueness.

From Lemma 1 and Theorem 1, we can see that $\hat{C}_\alpha^n$ does not fall into the trivial branch for $\beta \geqq \beta_1$. Then, how does $\hat{C}_\alpha^n$ behave when β is near zero? The following result for small β due to [7] is needed to answer this problem. (See also [2].)

Theorem 2. *Suppose that* (A-1)–(A-4) *hold. Then there exists some positive constant* β_0 *such that the problem* (1) *has a family of solutions* $(u_\beta(x;\alpha), v_\beta(x;\alpha))$ *for* $0<\beta<\beta_0$ *and as* $\beta \to 0$, *this family converges to the solution of the reduced problem of* (1). *Moreover there exists a positive constant* $\delta_0=\delta_0(\beta)$ *and* α_0 *such that* (1) *has no other solution in a* δ_0*-nbd. of* $(u_\beta(x;\alpha), v_\beta(x;\alpha))$ *in the* C^2*-topology for any* $\alpha \in (0, \alpha_0)$ *when* $\beta \in (0, \beta_0)$ *is fixed.*

Remark 3. When $\beta \in (0, \beta_0)$ is fixed and α tends to zero, $(u_\beta(x;\alpha), v_\beta(x;\alpha))$ converges to the solution of the shadow system.

§3. Global branching theorem. Before going into the last stage, we impose one more condition on the global branch of the shadow system, which is satisfied by several important models.

(A-5) There exists some positive constant β_2 such that $\mathrm{Sec}_\beta(\hat{C}_0^n)$ consists of two elements for $\beta<\beta_2$, where $\mathrm{Sec}_\beta(\hat{C}_0^n)$ denotes the β-section of $\hat{C}_0^n$.

Now we can show that $\hat{C}_\alpha^n$ coincides with the singular perturbation solutions for small β. More precisely, we have

Theorem 3 (Existence of the global branch and its asymptotic behavior as $\beta \to 0$). *Under the assumptions* (A-1)–(A-5), $\hat{C}_\alpha^n$ *exists globally with respect to* β *and coincides with the singular perturbation solutions in Theorem* 2 *when* β *is sufficiently small.*

Proof. First we take β_1 in Lemma 1 smaller than $\min\{\beta_0, \beta_2\}$, where β_0 and β_2 are the constants which appeared in Theorem 2 and (A-5), respectively. Then we take α sufficiently small such that for some fixed β with $\beta_1 \leqq \beta < \min\{\beta_0, \beta_2\}$, $\mathrm{Sec}_\beta(\hat{C}_\alpha^n)$ belongs to the interior of the δ_0-nbd. of the singular perturbation solution (this is possible because both $\mathrm{Sec}_\beta(\hat{C}_\alpha^n)$ and the singular perturbation solution converge to the same solutions $\mathrm{Sec}_\beta(\hat{C}_0^n)$ in (A-5) as $\alpha \to 0$). Local uniqueness in Theorem 2 implies that $\mathrm{Sec}_\beta(\hat{C}_\alpha^n)$ must coincide with the singular perturbation solution. This completes the proof.

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