42. *Poles of Instantons and Jumping Lines of Algebraic Vector Bundles on* $\mathbf{P}^3$

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Let E be an algebraic vector bundle of rank n on the complex 3-dimensional projective space $\boldsymbol{P}^3$ such that

(1) E has no global holomorphic sections,

(2) $c_1(E)=0$, $c_2(E)=k>0$, $c_3(E)=0$, where c_1, c_2 and c_3 denote the *Chern classes* of E, which are regarded as integers,

(3) for each general line L in $\boldsymbol{P}^3$, the restriction $E|_L$ is the trivial bundle of rank n on $L\cong\boldsymbol{P}^1$.

If $E|_L$ is not trivial, the line L is called a *jumping line* of E. These lines from an algebraic subset J of the *Grassmann variety* $\mathrm{Gr}(1,3)$ which parametrizes lines in $\boldsymbol{P}^3$.

In the case $n=2$, (1) and (2) imply that E is a stable bundle and (3) follows from them. Barth [2] has shown that in this case J is a divisor of degree $k=c_2(E)$ on $\mathrm{Gr}(1,3)$.

Our question is the following: When is E determined uniquely by the set J of its jumping lines?

For $n=2$ one has some affirmative answers by Barth [2] ($c_2(E)=1$) and Hartshorne [4] ($c_2(E)=2$). In the present article we shall state that this is true for all such bundles of any rank which come from "instantons".

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1. First of all we define the term instanton. Let P be a non-trivial real analytic principal bundle on the real 4-sphere $S=S^4$ with fibre $SU(n)$, called the *gauge group*. For a real analytic connection from ω on P, the corresponding curvature from Ω is given by $\Omega=d\omega+\frac{1}{2}[\omega,\omega]$, and it descends to S as a 2-form with values in the Lie algebra $\mathfrak{su}(n)$. The *self-dual* (resp. *anti-self-dual*) *Yang-Mills equation* is by definition the 1st order non-linear differential equation $*\Omega=\Omega$ (resp. $*\Omega=-\Omega$) for ω, where $*$ denotes the Hodge star operator on S. The difference between self-dual and anti-self-dual is a matter of orientation of S. So we choose and fix an orientation of S so that the 2nd Chern class $c_2(P)=k$ regarded as an integer is positive, and deal only with anti-self-dual equations.

A solution of the anti-self-dual Yang-Mills equation is called an *instanton solution* with *topological quantum number* $k=c_2(P)$, or simply a *k-instanton.* An $SU(n)$-connection is called *irreducible* if it does not come trivially from $SU(n-1)$. For two instanton solutions ω_1 and ω_2, they are said to be *gauge equivalent* if there exists a bundle automorphism $g: P \to P$ called a *gauge transformation* such that $\omega_1 = g^*\omega_2$. Atiyah-Ward [1] have shown that a k-instanton ω with gauge group $SU(n)$ corresponds injectively up to gauge equivalence to a rank n algebraic vector bundle on $\boldsymbol{P}^3$ with $c_1=c_3=0$ and $c_2=k$. To describe their transform we need a map $\pi: \boldsymbol{P}^3 \to S$. Let $\boldsymbol{H}$ be the Hamilton quaternion field. We identify $\boldsymbol{H}^2$ with $\boldsymbol{C}^4$. Then the natural projection $\boldsymbol{H}^2-\{0\} \to \boldsymbol{H}^2-\{0\}/\boldsymbol{H}^* = S$ induces a map $\pi: \boldsymbol{P}^3 \to S$ via this identification. Now let $\pi^*P^{\boldsymbol{C}}$ denote the complexified principal bundle of π^*P on $\boldsymbol{P}^3$ with fibre $SL(n, \boldsymbol{C})$. The pull-back $\pi^*\omega$ of an instanton ω can be viewed as a connection form of $\pi^*P^{\boldsymbol{C}}$ and induces an integrable almost complex structure on it. We denote by $E(\omega)$ the associated vector bundle of this complex analytic principal bundle $\pi^*P^{\boldsymbol{C}}$.

2. Next put $Gr=Gr(1,3)$. The flag variety $Fl=Fl(0,1,3)$ can be embedded in $Gr \times \boldsymbol{P}^3$ with natural projections $\alpha: Fl \to Gr$ and $\beta: Fl \to \boldsymbol{P}^3$. We denote by τ the algebraic correspondence $\alpha \circ \beta^{-1}$ from $\boldsymbol{P}^3$ to Gr. Then $\tau(\xi)$ is isomorphic to $\boldsymbol{P}^2$ for each point ξ in $\boldsymbol{P}^3$ and $\tau^{-1}(x)$ is isomorphic to $\boldsymbol{P}^1$ for each point x in Gr.

We choose a homogeneous coordinate $z=(z_0 : \cdots : z_5)$ of $\boldsymbol{P}^5$ so that the quadric Gr is determined by the equation $z_0^2 = z_1^2 + \cdots + z_5^2$. Then the real point set of Gr with respect to z is S. If we denote by $t_1, \cdots, t_5$ the real parts of $z_1, \cdots, z_5$ respectively, then S will be given as a subset of $\boldsymbol{R}^5$ by the equation $t_1^2 + \cdots + t_5^2 = 1$. Let us choose the coordinate z so that it also satisfies the following condition; the given orientation of S is compatible with that of $\boldsymbol{R}^5$ defined by the volume element $dt_1 \wedge \cdots \wedge dt_5$. In this way we obtain an embedding $\iota: S \to Gr$. Then Diagram 1 is commutative.

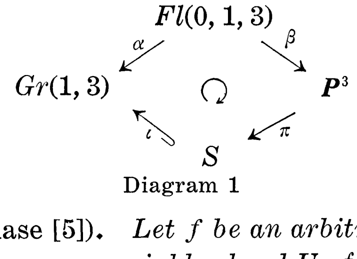

Diagram 1

Theorem 1 (Mulase [5]). *Let f be an arbitrary analytic form of type $(2,0)$ defined on an open neighborhood U of S in $Gr(1,3)$.*

(1) *If $f|_S = \iota^* f$ is an anti-self-dual 2-form on S, then the restriction $f|_{\tau(\xi)\cap U}$ of f vanishes identically on $\tau(\xi) \cap U$ for every point ξ in $\boldsymbol{P}^3$.*

(2) *If the restriction $f|_{\tau(\xi)\cap U}$ of f vanishes identically on $\tau(\xi) \cap U$*

for every point ξ in $\tau^{-1}(x)$ where x is an arbitrary point in S, then $f|_S$ is anti-self-dual.

Remark. This fact has been obtained independently by Belavin-Zakharov [3] in the case where f is a curvature form.

3. In what follows we do not distinguish between a vector bundle and its associated locally free sheaf of sections. For an instanton ω, it is clear that the restricted bundle $E(\omega)|_L$ of $E(\omega)$ to a general line L in $\boldsymbol{P}^3$ is analytically trivial. Since $c_1(E(\omega))=0$, L is a jumping line if and only if $H^0(L, E(\omega)\otimes\mathcal{O}_{\boldsymbol{P}^3}(-1)|_L)\neq 0$, where $\mathcal{O}_{\boldsymbol{P}^m}(-1)$ denotes the dual bundle of the hyperplane bundle $\mathcal{O}_{\boldsymbol{P}^m}(1)$ on $\boldsymbol{P}^m$.

Now let us consider an analytic sheaf $\mathcal{L}=R^1\alpha_*\beta^*(E(\omega)^\vee\otimes\mathcal{O}_{\boldsymbol{P}^3}(-1))$ on the Grassmann variety Gr, where $E(\omega)^\vee$ is the dual bundle of $E(\omega)$. Since α is a projection map of $\boldsymbol{P}^1$-bundle on Gr, $\mathcal{L}$ is a coherent sheaf. Then $J(\omega)=\operatorname{supp}\mathcal{L}$ is an analytic subset of Gr. By the Serre duality $H^0(L, E(\omega)\otimes\mathcal{O}_{\boldsymbol{P}^3}(-1)|_L)\cong H^1(L, E(\omega)^\vee\otimes\mathcal{O}_{\boldsymbol{P}^3}(-1)|_L)$, $\tau^{-1}(x)$ is a jumping line if and only if $x\in J(\omega)$. Using a locally free resolution of $\mathcal{L}$, we know that $J(\omega)$ is of codimension 1 everywhere in Gr. We define the *degree* of the divisor $J(\omega)$ by the 1st Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$. Then the degree of $J(\omega)$ is equal to $k=c_2(E(\omega))$. (Recall that $\operatorname{Pic}(Gr)\cong\boldsymbol{Z}$.)

An irreducible instanton corresponds to a simple vector bundle. By definition, a vector bundle is *simple* if it has no global endomorphism besides constant multiplications. A simple vector bundle on $\boldsymbol{P}^3$ which comes from an irreducible instanton has no global holomorphic sections. We deal with irreducible instantons from now on.

Let E be the associated vector bundle of P, i.e. $E=P\times_{SU(n)}\boldsymbol{C}^n$. The coherent sheaf $\tilde{E}_\omega=\alpha_*\beta^*E(\omega)|_{Gr-J(\omega)}$ is actually a vector bundle on $Gr-J(\omega)$ and the induced bundle $\iota^*\tilde{E}_\omega=\tilde{E}_\omega|_S$ on S is isomorphic to E.

We do not distinguish between a connection form defined on a principal bundle and a connection on the associated vector bundle. Since $Gr-J(\omega)$ is an affine algebraic manifold, it has affine coverings. If we choose such a suitable covering, ω can be continued analytically onto $Gr-J(\omega)$ as a (meromorphic) connection $\tilde{\omega}$ on $\tilde{E}_\omega$. This connection $\tilde{\omega}$ is algebraic, because ω has an algebraic character with respect to the natural real algebraic structure of S in Gr. For any point x in S, the Zariski open set $U_x=Gr-\tau(\tau^{-1}(x))$ of Gr is isomorphic to $\boldsymbol{C}^4$. Hence the restricted bundle $\tilde{E}_\omega|_{U_x-J(\omega)}$ is analytically trivial by the obstruction theory of Chern classes. Then $\tilde{\omega}$ can be represented as a meromorphic form of type (1, 0) on each $U_x-J(\omega)$, and has singularities along $J(\omega)$. By virtue of Theorem 1, however, we can show that $\tilde{\omega}$ is holomorphic everywhere on $Gr-J(\omega)$.

4. We can reconstruct the bundle $E(\omega)$ on $\boldsymbol{P}^3$ from the bundle $\tilde{E}_\omega$ on Gr by means of $\tilde{\omega}$. Careful observation of this reconstruction leads us to the following

Theorem 2. *Any simple vector bundle on P^3 of any rank which comes from an irreducible instanton is uniquely determined by the divisor on $Gr(1,3)$ of its jumping lines. More precisely, let $E(\omega_1)$ and $E(\omega_2)$ be simple vector bundles of the same rank on P^3 which come from irreducible instantons ω_1 and ω_2 respectively. Then $E(\omega_1)$ is isomorphic to $E(\omega_2)$ if and only if $J(\omega_1)=J(\omega_2)$ as a subset of $Gr(1,3)$.*

Corollary. *Any irreducible instanton solution with the gauge group $SU(n)$ is uniquely determined up to gauge equivalence by the location of the divisor of its poles (which can not be removed by any gauge transformation) in the complexified domain $Gr(1,3)$.*

Sketch of the proof of Theorem 2. Put $J=J(\omega_1)=J(\omega_2)$. Since the complex structure of $E(\omega_1)$ (resp. $E(\omega_2)$) is given by the connection $\pi^*\omega_1$ (resp. $\pi^*\omega_2$), we have an isomorphism $H^0(L, E(\omega_1)|_L)\cong H^0(L, E(\omega_2)|_L)$ depending real analytically on L, where L is a line corresponds to a point in $Gr-J$. This isomorphism induces a real analytic isomorphism of two bundles $\tilde{E}_{\omega_1}=\alpha_*\beta^*E(\omega_1)|_{Gr-J}$ and $\tilde{E}_{\omega_2}=\alpha_*\beta^*E(\omega_2)|_{Gr-J}$ on $Gr-J$. Then $\tilde{E}_{\omega_1}$ is isomorphic to $\tilde{E}_{\omega_2}$ in the complex analytic sense, because $Gr-J$ is a Stein manifold. We denote the bundle $\tilde{E}_{\omega_1}\cong\tilde{E}_{\omega_2}$ simply by $\tilde{E}$. The analytically continued instantons $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are viewed as connections of the same bundle $\tilde{E}$ on $Gr-J$. Let $\bar{\alpha}^*\tilde{\omega}_1$ (resp. $\bar{\alpha}^*\tilde{\omega}_2$) denote the pull-back of the connection $\tilde{\omega}_1$ (resp. $\tilde{\omega}_2$) via the restricted map $\bar{\alpha}=\alpha|_{Fl-\alpha^{-1}(J)}$. Then they are holomorphic connections of the bundle $\alpha^*\tilde{E}|_{Fl-\alpha^{-1}(J)}=\bar{\alpha}^*\tilde{E}$. Since α is locally a projection map, $\alpha^*\tilde{E}_{\omega_i}|_{Fl-\alpha^{-1}(J)}$ is isomorphic to $\beta^*E(\omega_i)|_{Fl-\alpha^{-1}(J)}$ $(i=1,2)$. Hence we have an isomorphism $\beta^*E(\omega_1)|_{Fl-\alpha^{-1}(J)}\cong\beta^*E(\omega_2)|_{Fl-\alpha^{-1}(J)}$. Put $\bar{\beta}=\beta|_{Fl-\alpha^{-1}(J)}$. Theorem 1 asserts that the restriction $\bar{\alpha}^*\tilde{E}|_{\bar{\beta}^{-1}(\xi)}$ of $\bar{\alpha}^*\tilde{E}$ to each fibre $\bar{\beta}^{-1}(\xi)$ of $\bar{\beta}$ is the trivial bundle with flat connections $\bar{\alpha}^*\tilde{\omega}_1|_{\bar{\beta}^{-1}(\xi)}$ and $\bar{\alpha}^*\tilde{\omega}_2|_{\bar{\beta}^{-1}(\xi)}$.

Now let us denote by $E_{\mathrm{flat}}(\omega_1)$ (resp. $E_{\mathrm{flat}}(\omega_2)$) the vector bundle on P^3 whose fibre on $\xi\in P^3$ is the vector space of flat sections of $\beta^*E(\omega_1)|_{\bar{\beta}^{-1}(\xi)}$ with respect to the flat connection $\bar{\alpha}^*\tilde{\omega}_1|_{\bar{\beta}^{-1}(\xi)}$ (resp. $\bar{\alpha}^*\tilde{\omega}_2|_{\bar{\beta}^{-1}(\xi)}$). If one observes these constructions carefully, one will obtain isomorphisms $E_{\mathrm{flat}}(\omega_1)\cong E(\omega_1)$ and $E_{\mathrm{flat}}(\omega_2)\cong E(\omega_2)$.

To compare $E_{\mathrm{flat}}(\omega_1)$ with $E_{\mathrm{flat}}(\omega_2)$, we introduce a suitable open covering of P^3. Set $U_\xi=\beta(\bar{\alpha}^{-1}(\tau(\xi))-\bar{\beta}^{-1}(\xi))$. U_ξ is an open dense subset of P^3. Then there exists a finite set $\{\xi_0,\xi_1,\cdots,\xi_\sigma\}$ of points in P^3 such that (i) $\bigcup_{\mu=1}^{\sigma}U_{\xi_\mu}=P^3$, (ii) the line passing through ξ_0 and ξ_μ does not corresponds to any point in J, for $\mu=1,2,\cdots,\sigma$. The bundles $E_{\mathrm{flat}}(\omega_1)$ and $E_{\mathrm{flat}}(\omega_2)$ are both trivial on each U_{ξ_μ}, $\mu=1,2,\cdots,\sigma$. Using $\tilde{\omega}_1$ (resp. $\tilde{\omega}_2$), we can choose canonically the local trivialization and the system of transition functions of $E_{\mathrm{flat}}(\omega_1)$ (resp. $E_{\mathrm{flat}}(\omega_2)$) associated with the covering $\bigcup_{\mu=1}^{\sigma}U_{\xi_\mu}$. One can see that these two systems of transition functions of $E_{\mathrm{flat}}(\omega_1)$ and $E_{\mathrm{flat}}(\omega_2)$ are equivalent.

Hence one obtains an isomorphism $E(\omega_1)\cong E(\omega_2)$.

The converse is obvious.

5. **Further remarks.** (i) We can show by Theorem 2 that $J(\omega)$ is a reduced divisor of Gr for any instanton ω. Hence, for any generic point x in $J(\omega)$, $E(\omega)|_{\tau^{-1}(x)}$ is isomorphic to $\mathcal{O}_{P^1}(1)\oplus\mathcal{O}_{P^1}(-1)\oplus\mathcal{O}_{P^1}^{\oplus(n-2)}$, where n is the rank of $E(\omega)$.

(ii) Let ω be a k-instanton solution with gauge group $SU(3)$. If ω is not irreducible, then $E(\omega)$ is an extension of certain simple vector bundle of rank 2 by the trivial line bundle on $\boldsymbol{P}^3$. In this case $E(\omega)$ is not determined by the divisor of its jumping lines.

(iii) Atiyah-Ward [1] observed that anti-self-dual Yang-Mills equations have algebraic characters. Hence we can start with C^∞-principal bundles and C^∞-connections instead of real analytic ones.

(iv) For a k-instanton ω with gauge group $SU(n)$, we have a residue type formula

$$\frac{1}{4\pi^2}\int_S \operatorname{trace} \Omega\wedge\Omega=\deg J(\omega)=k.$$

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