35. On Plane Rational Curves

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1. The logarithmic Kodaira dimension $\bar{\kappa}(V)$ of an algebraic variety V, introduced by S. Iitaka [1], plays an important role in the study of algebraic varieties. In this note, we announce some results concerning the logarithmic Kodaira dimension $\bar{\kappa}(\boldsymbol{P}^2-C)$, where C is an irreducible curve on $\boldsymbol{P}^2$. Here we consider only rational curves C which satisfy

(A) C has one singular point,

or

(B) C has two singular points, which are cusps.

Any singular point P of C is said to be a *cusp*, if $\mu^{-1}(P)$ is one point, where $\mu: C^* \to C$ is the reduction of singularities. By the recent result of I. Wakabayashi [3], if the curve C does not satisfy the above conditions, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$ or C is a non-singular curve with degree $\leqq 3$. Details will appear elsewhere. The author would like to express his hearty thanks to Prof. S. Iitaka for suggesting this problem and giving valuable advice.

2. In this note, we use the following convention. Let (X_0, X_1, X_2) denote the system of homogeneous coordinates on $\boldsymbol{P}^2$ and $x=X_1/X_0$, $y=X_2/X_0$. Since we do not consider $V_+(X_0)$, we say that the irreducible curve $V_+(X_0^n f(X_1/X_0, X_2/X_0))$ is defined by f, where f is an irreducible polynomial of degree n. Moreover let C denote a rational curve of degree n on $\boldsymbol{P}^2$. In the case of (A), let P be the singular point of C, and put $e=\mathrm{mult}_P(C)$. Denote by N the number of points of $\mu^{-1}(P)$. In the case of (B), let P, Q denote two singular points of C, and $(e_1, \cdots, e_p)$ [resp. $(m_1, \cdots, m_q)$] the sequence of the multiplicities of all the (infinitely near) singular points of P [resp. Q].

3. Put $k=e$ in the case of (A), and $k=\max\{e_p, m_q\}$ in the case of (B), respectively. Then we have the following result.

Proposition 1. *In both cases, if $n \geqq 3k$, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

In the case of (B), define

$$M=\sum_{i=1}^{p}(e_i-1)+\sum_{j=1}^{q}(m_j-1)+d+6-3n,$$

where $d=\min\{e_p, m_q\}$. Then

Proposition 2. *If $M>0$, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

However, it seems difficult to construct rational curves satisfying the hypothesis of Propositions 1 or 2.

Now we shall study curves of the case (A). First note that there exist rational curves with $e=2$, $n\leqq 5$ as follows.

Proposition 3. *Suppose that C satisfies* (A) *with $e=2$, $n\leqq 5$. Then it is projectively equivalent to one of the curves defined by the following polynomials.*

n	cusp case i.e. $N=1$	non-cusp case i.e. $N=2$
3	y^2-x^3	$xy-x^3-y^3$
4	$(y-x^2)^2+xy^3$	$(y-x^2)^2+tx^2y^2+xy^3$, $t\in \boldsymbol{C}-\{0\}$
5	$(y-x^2)(y-x^2+2xy^2)+y^5$	$(y-x^2)(y-x^2+ty^2-tx^2y+2xy^2)+y^5$, $t\in \boldsymbol{C}-\{0\}$

Let C_t be the curve in the non-cusp case with $n=4, 5$. Then C_t and C_s are projectively equivalent if and only if $t^3=s^3$, i.e., t^3 is the projective invariant. In the cusp case, we have $\bar{\kappa}(\boldsymbol{P}^2-C)=-\infty$, while in the non-cusp case, $\bar{\kappa}(\boldsymbol{P}^2-C)=0$. More precisely, $\bar{P}_m(\boldsymbol{P}^2-C)=1$ for all $m\geqq 1$, where $\bar{P}_m$ denote the logarithmic m-genera (*see* [1]). *Thus, these $\boldsymbol{P}^2-C$ are logarithmic $K3$ surfaces* (*see* [2]).

But in the case where $n=6$, we have a new phenomenon.

Theorem. *Suppose that C satisfies* (A) *with $e=2$, $n=6$. Then P cannot be a cusp, and C is defined by $f(x, y)=y^2+\sum_{i+j=3}^{6} c_{ij}x^iy^j$ such that the c_{ij} are determined as follows; let α and β be complex numbers satisfying $4\alpha^2-30\alpha+55=0$, $\beta^3+(\alpha-3)(\alpha-4)=0$. Put $c_1=2(8\alpha-33)\beta^2$, $c_2=(3\alpha-23/2)\beta$, $c_3=(-21\alpha+159/2)\beta$, $c_4=-4\alpha+13$, $c_5=(-16\alpha+67)\beta^2$, $c_6=(24\alpha-92)\beta$, and further put $c_{30}=0$, $c_{21}=-2$, $c_{12}=c_1$, $c_{03}=c_2$, $c_{40}=1$, $c_{31}=-2c_1$, $c_{22}=c_3$, $c_{13}=c_4$, $c_{04}=c_5$, $c_{50}=c_1$, $c_{41}=-3c_2-2c_3$, $c_{32}=-2c_4-2$, $c_{23}=-2c_1-2c_5$, $c_{14}=c_6$, $c_{05}=-c_4-2$, $c_{60}=2c_2+c_3$, $c_{51}=c_4+2$, $c_{42}=2c_1+c_5$, $c_{33}=-c_6$, $c_{24}=c_4+3$, $c_{15}=c_1$, $c_{06}=-3c_2-c_3-c_6$. Further, three curves defined by three different values β for the given α are projectively equivalent to each other.*

Remark. Since we can apply Proposition 1 to the curve C in the above theorem, we have $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.

4. It seems interesting to classify plane curves C by $\bar{\kappa}=\bar{\kappa}(\boldsymbol{P}^2-C)$. Referring to [3], we have only to deal with the cases (A) and (B) in the first section. In the case of degree $n\leqq 5$, the result is as follows:

The case (A). First, note that if $n=5$ and $e=3$, then $N\geqq 2$. Given any triple (n, e, N) with $4\geqq n-1\geqq e\geqq N$, there exist curves C of the case (A) such that the degree of C is equal to n, $\text{mult}_P(C)=e$, and $\mu^{-1}(P)$ consists of N points, except for $(n, e, N)=(5, 3, 1)$. The classification is shown in Table I. By this table, we conclude that: $\bar{\kappa}=-\infty \Leftrightarrow N=1$; $\bar{\kappa}=0 \Leftrightarrow N=2$; and $\bar{\kappa}=1 \Leftrightarrow N=3$ or 4.

Table I

e	N	$\bar{\kappa}$
2	1	$-\infty$
	2	0
3	1	$-\infty$
	2	0
	3	1
4	1	$-\infty$
	2	0
	3,4	1

The case (B). The pair of the sequences of multiplicities $((e_1, \cdots, e_p), (m_1, \cdots, m_q))$, $p \leqq q$, is said to be the *type of cusps of* C.

(1) If $n=4$, then the type of cusps of C is ((2), (2, 2)), and $\bar{\kappa}=1$. Moreover, C is projectively equivalent to the curve defined by $f(x, y) = (y-x^2)^2 + x^3y$.

(2) If $n=5$, we have the following table:

Table II

	type of cusps	$\bar{\kappa}$
(i)	((3), (2, 2, 2))	2
(ii)	((3, 2), (2, 2))	1
(iii)	((2, 2), (2, 2, 2, 2))	2

Note that there do not exist curves with types of cusps which are not listed in Table II. Moreover, the curves of (i), (ii), and (iii) are projectively equivalent to the curves defined by the following polynomials, respectively:

(i) $f(x, y) = y^3 + x^4 + x^3y - (1/2)x^2y^2 - (1/4)x^5 + (1/16)x^4y$.

(ii) $f(x, y) = y^3 - x^5$, or $y^3 + x^2y^2 - x^5 + (1/4)x^4y$.

(iii) $f(x, y) = (y-x^2)^2 + 6xy^2 + y^3 - 12x^3y + 6x^2y^2 + 8xy^3 + 6x^5 - 7x^4y - 8x^3y^2 + 16x^2y^3$.

Remark. In the cases (i), (ii) in Table II, the values M defined in front of Proposition 2 are -2.

5. **Proposition 4.** *Suppose that $C - C \cap \Delta$ is isomorphic to A^1 for an irreducible conic Δ. Then, in the case where $3 \leqq n \leqq 6$, C is projectively equivalent to the curve defined by $f(x, y) = (y-x^2)(y-x^2+2xy^2) + y^5$, which is the curve in Proposition* 3.

Remark. Even if $\bar{\kappa}(\boldsymbol{P}^2 - C) = -\infty$, we have a non-discrete projective invariant. In fact, let C be a curve defined by $f_t(x, y) = y^4 + x^5 + x^3y^2 + tx^2y^3$, $t \in \boldsymbol{C}$. Then, the projective invariant of C is t^2.

Finally, we ask the following

Problem. Suppose that a rational curve C has only one singular point which is a cusp. Then, is it true that $\bar{\kappa}(\boldsymbol{P}^2 - C) = -\infty$?

References

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[3] I. Wakabayashi: On the logarithmic Kodaira dimension of the complement of a curve in $\boldsymbol{P}^2$. Ibid., **54**A, 157–162 (1978).