4. *Asymptotic Equivalence of Dynamical Systems*

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In [1] the author generalizes the notion of asymptotic equivalence and attempts to prove theorems which are related to results in [2, Chapter IX, Section 4]. Unfortunately, this definition of asymptotic equivalence is inadequate to guarantee the stated results. (This is not noted in the review of [1] in Mathematical Reviews, MR 45 #7211.) In this paper we present a counter example to two of the theorems in [1] and redefine "asymptotic equivalence" in such a manner as to validate the theorems to which we provide a counter example. In fact, stronger results, as would be expected by a more restrictive definition, are proved.

Throughout this paper X will denote a locally compact metric space with metric d, R the reals, and R^+ the nonnegative reals. If $M \subset X$ and $a>0$, then $K(M, a)$ will denote the set $\{x: d(x, M)<a\}$.

A dynamical system on X is a continuous mapping $\pi: X \times R \to X$ such that

(i) $\pi(x, 0)=x$ for all $x \in X$,

(ii) $\pi(\pi(x, s), t)=\pi(x, s+t)$ for all $x \in X$ and $s, t \in R$.

Let π_i $(i=1,2)$ be dynamical systems on X and $x \in X$. Then $L_i(x)$ and $J_i(x)$ will denote the positive limit set of x and the positive prolongational limit set of x, respectively, with respect to π_i. A compact subset M of X is called

(i) a weak attractor of π_i, if there exists an $a>0$ such that $L_i(x) \cap M \neq \phi$ for every $x \in K(M, a)$,

(ii) an attractor of π_i, if there exists an $a>0$ such that $\phi \neq L_i(x) \subset M$ for every $x \in K(M, a)$,

(iii) a uniform attractor of π_i, if there exists an $a>0$ such that $\phi \neq J_i(x) \subset M$ for every $x \in K(M, a)$,

(iv) stable with respect to π_i, if for any $a>0$ there exists $b>0$ such that $\pi_i(K(M, b), R^+) \subset K(M, a)$,

(v) eventually stable with respect to π_i, if for any $a>0$ there exist $b>0$ and $T>0$ such that $\pi_i(K(M, b), [T, \infty)) \subset K(M, a)$,

(vi) weakly asymptotically stable with respect to π_i, if M is eventually stable and a weak attractor with respect to π_i,

(vii) asymptotically stable with respect to π_i, if M is stable and an attractor with respect to π_i,

(viii) positively invariant with respect to π_i, if $\pi_i(M, R^+)=M$.

In [1] the dynamical systems π_1 and π_2 are said to be asymptotically equivalent (henceforth called K-asymptotically equivalent) on a subset S of X if $d(\pi_1(x,t), \pi_2(y,t))\to 0$ as $t\to\infty$ for every $x, y \in S$.

Two of the principle results of [1] are (Theorems 3.7 and 3.8).

Theorem A. *Let*

(1) *X be a compact metric space,*

(2) *S be a nonempty open subset of X,*

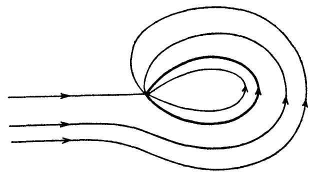
(3) *M be a compact subset of S which is stable with respect to π_1,*

(4) *π_1 be K-asymptotically equivalent to π_2 on S.*

Then M is eventually stable with respect to π_2.

Theorem B. *Let X, S, and M be as in Theorem* A. *If M is asymptotically stable with respect to π_1 and if π_1 is K-asymptotically equivalent to π_2, then M is weakly asymptotically stable with respect to π_2.*

Unfortunately, both of these results are false. Let ρ_1 be a planar dynamical system in which the origin is globally asymptotically stable and let ρ_2 be the planar dynamical system indicated in the following diagram. (The critical point is the origin.)



Note that the origin is not eventually stable with respect to ρ_2. Evidently ρ_1 and ρ_2 are K-asymptotically equivalent. Now let π_1 and π_2 be the dynamical systems induced by ρ_1 and ρ_2 on the one point compactification X of the plane. Set $S=X-\{\infty\}$. Then π_1 and π_2 are K-asymptotically equivalent on S. The image of the origin is asymptotically stable with respect to π_1, but is neither eventually stable nor weakly asymptotically stable with respect to π_2.

The error in the proof of Theorem A is that it is erroneously assumed that $d(\pi_1(x,t), \pi_2(y,t))\to 0$ uniformly as $t\to\infty$. We will now give definitions for two types of asymptotic equivalence: one weaker than K-asymptotic equivalence and the other incorporating the uniform behavior indicated above.

Definition 1. (i) π_1 and π_2 are said to be asymptotically equivalent (denoted by $\pi_1 - \pi_2$) if $d(\pi_1(x,t), \pi_2(x,t))\to 0$ as $t\to\infty$ for every $x \in X$.

(ii) π_1 and π_2 are said to be uniformly asymptotically equivalent (denoted by $\pi_1 -_u \pi_2$) if for any compact subset N of X and $a>0$ there is a $T>0$ such that $d(\pi_1(x,t), \pi_2(y,t))<a$ whenever $t\geq T$ and $x, y \in N$.

Evidently if $\pi_1 -_u \pi_2$, then $\pi_1 - \pi_2$.

Lemma 2. *If a compact subset M of X is an attractor of π_1 and $\pi_1 - \pi_2$, then M is an attractor of π_2.*

Proof. Let U be a compact neighborhood of M such that $\phi \neq L_1(x) \cap M$ for all $x \in U$. Then $d(M, \pi_2(x, t)) \leq d(M, \pi_1(x, t)) + d(\pi_1(x, t), \pi_2(x, t)) \to 0$ as $t \to \infty$ for every $x \in U$, since M is an attractor of π_1 and $\pi_1 - \pi_2$. It easily follows that $L_2(x) \subset M$ for every $x \in U$. M is an attractor of π_2.

Lemma 3. *Let M be a positively invariant, compact subset of X. Then M is a uniform attractor of π_1 if and only if there is a neighborhood U of M such that for any neighborhood V of M there exists $T > 0$ such that $\pi_1(U, [T, \infty)) \subset V$.*

Proof. Let M be a uniform attractor. Then there is a compact neighborhood U of M such that $\phi \neq J_i(x) \subset M$ for all $x \in U$. Let W be any compact neighborhood of M. Suppose that there is no T such that $\pi_i(U, [T, \infty)) \subset W$. Then there are a sequence $\{x_j\}$ in U and a sequence $\{t_j\}$ in R^+ such that $x_j \to x$ for some $x \in M$, $t_j \to \infty$ or $t_j \to t$ for some $t \in R^+$, and $\pi_i(x_j, t_j) \in \partial W$. Since ∂W is compact, $\{\pi_i(x_j, t_j)\}$ has an accumulation point $y \in \partial W$. If $t_j \to \infty$, then $y \in J_i(x)$. This is impossible because $J_i(x) \subset M$ and $y \in \partial W \subset X$-interior $W \subset X - M$. If $t_j \to t$, then $\pi_i(x, t) \in \partial W$. But M is invariant so that $\pi_i(x, t) \in M$. This is impossible because $\partial W \subset X - M$. These contradictions imply that there is a T such that $\pi_i(U, [T, \infty)) \subset W$. It easily follows that if V is any neighborhood of M, then there is a $T > 0$ such that $\pi_i(U, [T, \infty)) \subset V$. The converse is easily verified.

Lemma 4. *Let a compact subset M of X be positively invariant with respect to both π_1 and π_2, stable with respect to π_1, and $\pi_1 -_u \pi_2$. Then M is stable with respect to π_2.*

Proof. We will first show that M is eventually stable with respect to π_2. Let $a > 0$. Then there exists $b > 0$ such that $\overline{K(M, b)}$ is compact and $\pi_1(\overline{K(M, b)}, R^+) \subset K(M, a/2)$. Since $\pi_1 -_u \pi_2$ there is a $T > 0$ such that $d(\pi_1(x, t), \pi_2(x, t)) < a/2$ for all $x \in \overline{K(M, b)}$ and $t \geq T$. It follows directly $\pi_2(x, t) \subset K(M, a)$ for all $x \in K(K, b)$ and $t \geq T$. Hence, M is eventually stable with respect to π_2. We will now show that if V is a compact neighborhood of M, then there is a neighborhood U of M such that $\pi_2(x, R^+) \subset V$ for every $x \in U$. Suppose the contrary. Then there are a sequence $x_i \to x \in M$ and a sequence $t_i \in R^+$ such that $\pi_1(x_i, t_i) \in \partial V$. Since M is eventually stable with respect to π_2, there are a neighborhood W of M and a $T > 0$ such that $\pi_2(W, t) \subset V$ for every $t \geq T$. Hence, eventually $t_i \leq T$. Without loss of generality we may assume that $t_i \to t \leq T$. Then $\pi_2(x, t) \leftarrow \pi_2(x_j, t_j) \in \partial V$. This is impossible since ∂V is compact, M positively invariant, and $\partial V \subset X$-interior $V \subset X - M$. Hence, there is a neighborhood U of M such that $\pi_2(x, R^+) \subset$ interior V for every $x \in U$. M is stable with respect to π_2.

Theorem 5. *Let a compact subset M of X be positively invariant with respect to both π_1 and π_2, asymptotically stable with respect to π_1, and $\pi_1 -_u \pi_2$. Then M is asymptotically stable with respect to π_2.*

Proof. If M is asymptotically stable, then M is a uniform attractor [3, Theorem 2.11.37]. The desired result now follows from Lemmas 2 and 4.

There appears to be no readily verifiable criterion for determining whether two dynamical systems are asymptotically equivalent. In [1, Proposition 3.3] the following is stated.

Proposition C. *Let X be locally compact and S a nonempty subset of X such that $L_1(x)$ and $L_2(x)$ are both nonempty and compact for every $x \in X$. Then π_1 is k-asymptotically equivalent to π_2 if and only if $L_1(x) \cap L_2(y) \neq \phi$ for every $x, y \in S$.*

Unfortunately, the "if" part of Proposition C is false. Consider periodic dynamical system π_1 on the circle $|z|=1$ which has period 1. Set $\pi_2 = \pi_1$. Then all limit sets coincide. Let x and y be distinct points on the circle. For each positive integer n we have $x = \pi_1(x, n) = \pi_2(x, n)$ and $y = \pi_1(y, n) = \pi_2(y, n)$ so that $d(\pi_1(x, n), \pi_2(y, n) = d(x, y) \neq 0$. Therefore $d(\pi_1(x, t), \pi_2(y, t) \not\to 0$ as $t \to \infty$. Hence π_1 and π_2 are not k-asymptotically equivalent.

References

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