34. *Family of Varieties Dominated by a Variety*

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§1. **Introduction.** Let k be an algebraically closed field of characteristic zero. We assume all varieties are defined over k. In this note we prove a finiteness theorem for the isomorphism classes of canonically polarized varieties dominated by any given variety.

It was proved by Severi that only finitely many (up to isomorphism) curves of genus $\geqq 2$ can be dominated by a given variety over any algebraically closed field (cf. [16]).

As a generalization of this result in higher dimensional cases over k, we shall prove the following

Main Theorem. *Only finitely many (up to isomorphism) canonically polarized varieties can be dominated by a given variety.*

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§2. **Statement of results.** We fix our notation. In this note, by a "variety" we shall always mean a proper integral algebraic space over k. A non-singular variety will be said to be canonically polarized if the canonical invertible sheaf is ample.

We denote by Θ_X, Ω_X^n and K_X the sheaf of regular tangent vectors, the sheaf of differential n-forms and the canonical invertible sheaf on a non-singular variety X, respectively.

Let X be a variety. $\mathcal{F}(X)$ denotes the set $\{(f, Y)\}$ of pairs (f, Y) each of which consists of a projective non-singular variety Y and a surjective morphism $f: X \to Y$. $(f_1, Y_1), (f_2, Y_2) \in \mathcal{F}(X)$ are said to be isomorphic to each other (or isomorphic in the strong sense) if there is an isomorphism $g: Y_1 \to Y_2$ (or $g \circ f_1 = f_2$).

$\mathcal{F}^a(X)$ denotes the subset of $\mathcal{F}(X)$ consisting of $\{(f, Y)\}$ with canonically polarized varieties $\{Y\}$.

Now our Main Theorem can be stated as follows:

Theorem 1. *$\mathcal{F}^a(X)$ is finite up to isomorphism in the strong sense.*

We can also show the following

Theorem 2. *$\mathcal{F}(X)$ is at most countable up to isomorphism.*

Shortly speaking at most countable (up to isomorphism) non-singular projective varieties can be dominated by X.

Remark 2.1. Let $\mathcal{F}^1(X)$ be a subset of $\mathcal{F}(X)$ consisting of $\{(f, Y)\}$ with $H^0(Y, f^*\Theta_Y)=0$. Then $\mathcal{F}^1(X)$ is at most countable up to isomorphism in the strong sense (see Horikawa [7]). Note that $H^0(Y, f^*\Theta_Y)=0$ if a general fibre of f is connected and $H^0(Y, \Theta_Y)=0$.

Remark 2.2. Let M be a compact integral Kähler space in the sense of Fujiki [2]. $\mathcal{F}^k(M)$ denotes the set $\{(f, N)\}$ of pairs (f, N) each of which consists of a compact Kähler manifold N and *a* surjective morphism $f: M \to N$. Then $\mathcal{F}^k(M)$ is at most countable up to isomorphism (see Fujiki [3]).

We have the following result of varieties of general type. For a given variety X, fix a sufficiently large number $L \geqq 2 \dim X+1$. Let $A(X)$ denote the set $\{(f, Y)\}$ of pairs (f, Y) each of which consists of a closed non-singular subvariety of $\boldsymbol{P}^L$ and a surjective morphism $f: X \to Y$. For a positive integer m, let $B_m(X)$ be a subset of $A(X)$ which satisfies the following condition.

1) There exists an effective divisor linearly equivalent to $mK_Y - H_Y$ where H_Y denotes a hyperplane section of Y. Note that a) any projective non-singular variety Y with $\dim Y \leqq \dim X$ can be embedded in $\boldsymbol{P}^L$, b) $Y \in A(X)$ is of general type if and only if $Y \in B_m(X)$ for a positive integer m.

Proposition 3.1. *The above $B_m(X)$ is finite up to isomorphism in the strong sense for each m.*

Proposition 3.2. *Let X be a non-singular projective variety. Let $B(X)$ be a subset of $A(X)$ which satisfies one of the following two conditions.*

1) *The Hilbert polynomial $\chi(X, m(H_X+f^*H_Y))$ are equal to a fixed polynomial $P(m)$ where H_X is a fixed very ample invertible sheaf on X.*

2) *$h^0(X, m(H_X+f^*H_Y)) \leqq P(m)$ for $m \geqq m_0$ where $P(m)$ is a fixed polynomial, m_0 is a fixed integer and H_X-K_X is ample.*

Then $B(X)$ is finite up to isomorphism.

Proposition 3.3. *Let X be a variety, L_X a fixed locally free coherent sheaf on X. Let $C(X, L_X)$ be a subset of $A(X)$ which satisfies one of the following two conditions.*

1) *There exists a non-zero homomorphism $f^*H_Y \to L_X$ for any $(f, Y) \in C(X, L_X)$.*

2) *There is a linear projection p which makes the following diagram commutative:*

$$\begin{array}{ccl} X & \overset{\cdots}{\longrightarrow} & \operatorname{Proj} S\Gamma(X, L_X) = \boldsymbol{P}(\Gamma(X, L_X)) \ (\operatorname{rank} L_X = 1) \\ f \downarrow & & \downarrow p \\ Y & \hookrightarrow & \boldsymbol{P}^L \end{array}$$

for any $(f, Y) \in C(X, L_X)$.

Then $C(X, L_X)$ is finite up to isomorphism.

Proposition 3.4. *Let X be a projective non-singular variety, L_X a locally free coherent sheaf on X. Let $\mathcal{G}(X, L_X)$ be a subset of $\mathcal{F}(X)$ satisfying the following two conditions.*

1) For any $(f, Y) \in \mathcal{G}(X, L_X)$ and a fixed integer $t>0$, there exists an invertible sheaf D_Y on Y such that K_Y+D_Y and $(t-1)K_Y+tD_Y$ are ample.

2) There exists a non-zero homomorphism $f^*(K_Y+D_Y)\to\Omega_X^n\otimes L_X$ where $n=\dim Y$, for any $(f, Y) \in \mathcal{G}(X, L_X)$.

Then $\mathcal{G}(X, L_X)$ is finite up to isomorphism.

Remark 4. 1) If X is a proper non-reduced algebraic space over $\boldsymbol{C}$, $\mathcal{F}^a(X)$ is finite up to isomorphism but not always finite in the strong sense.

2) Let k be a field of characteristic zero. Even when all varieties are defined over k, Main Theorem holds.

§3. Proof of Theorem 1. We now give an outline of the proof of Theorem 1. By Moishezon [13], we have a surjective morphism from a projective non-singular variety $\tilde{X}$ onto X. Hence, replacing X by $\tilde{X}$, we may assume X to be projective and non-singular.

Lemma 1. *There are only finitely many polynomials of the form $\chi(Y, mK_Y)=\Sigma(-1)^p \dim H^p(Y, mK_Y)$, where $Y \in \mathcal{F}^a(X)$.*

Proof. By the Kodaira vanishing theorem, we have

$$\chi(Y, mK_Y)=h^0(Y, mK_Y) \qquad \text{for } m\geqq 2.$$

We have the following inequality

$$0\leqq h^0(Y, mK_Y)\leqq h^0(X, S^m\Omega_X^n),$$

where S^m denotes m-th symmetric product and $n=\dim Y\leqq\dim X$. The lemma follows from the fact that a polynomial of degree n is determined by its $(n+1)$ values. Q.E.D.

By Lemma 1 and Matsusaka [12], we have a positive integer m_0 such that m_0K_Y is very ample for every $Y \in \mathcal{F}^a(X)$. Hence there is a projective space $\boldsymbol{P}^N$ in which every $Y \in \mathcal{F}^a(X)$ can be embedded in such a way that $m_0K_Y\cong H_Y$, where H_Y denotes a hyperplane section.

Next we consider the graph Γ_f of $f: X\to Y, (Y \in \mathcal{F}^a(X))$. Then $\Gamma_f\subset X\times\boldsymbol{P}^N$. Take a very ample invertible sheaf H_X on X such that $H_X\otimes K_X^{-1}$ is ample. Note that the invertible sheaf $H_X\otimes H_{\boldsymbol{P}^N}\cong H$ on $X\times\boldsymbol{P}^N$ is very ample.

Lemma 2. *There are only finitely many polynomials of the form $\chi(\Gamma_f, mH_{\Gamma_f})$, where $(f, Y) \in \mathcal{F}^a(X)$.*

Proof. By the Kodaira vanishing theorem, we have

$$\chi(\Gamma_f, mH_{\Gamma_f})=\chi(X, m(H_X+f^*m_0K_Y))=h^0(X, m(H_X+f^*m_0K_Y)) \qquad \text{for } m\geqq 1.$$

We have the following inequality

$$0\leqq h^0(X, m(H_X+f^*m_0K_Y))\leqq h^0(X, H_X^m\otimes S^{mm_0}\Omega_X^n).$$

In a similar manner as in the proof of Lemma 1, we can complete the proof. Q.E.D.

We parametrize morphisms from X into $\boldsymbol{P}^N$ by an open subscheme $\mathrm{Hom}(X, \boldsymbol{P}^N)$ of the Hilbert scheme $\mathrm{Hilb}_{X\times \boldsymbol{P}^N}$. For our purpose, we have only to consider the components $\mathrm{Hom}^{p(m)}(X, \boldsymbol{P}^N)$ where $p(m)$ are the polynomials determined by Lemma 2. Let $\mathrm{Hom}^*(X, \boldsymbol{P}^N)$ be the union of them. By Grothendieck [6], the Hilbert scheme $\mathrm{Hilb}^{p(m)}_{X\times \boldsymbol{P}^N}$ is projective. Hence $\mathrm{Hom}^*(X, \boldsymbol{P}^N)$ is a Noetherian scheme. We use repeatedly the fact that if $j: W\to W'$ is an immersion and W' is a Noetherian scheme, then W is also a Noetherian scheme. We have the natural commutative triangle

$$\begin{array}{ccc} X\times \mathrm{Hom}^*(X,\boldsymbol{P}^N) & \longrightarrow & \boldsymbol{P}^N\times \mathrm{Hom}^*(X,\boldsymbol{P}^N) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}^*(X,\boldsymbol{P}^N). & \end{array}$$

Let Γ be a universal member of the Hilbert functor. We denote by Ξ the image of $\Gamma_{(\mathrm{Hom}^*(X,\boldsymbol{P}^N))}$ given by the projection

$$X\times \boldsymbol{P}^N\times \mathrm{Hom}^*(X,\boldsymbol{P}^N)\to \boldsymbol{P}^N\times \mathrm{Hom}^*(X,\boldsymbol{P}^N).$$

There is a flattening stratification on $\mathrm{Hom}^*(X, \boldsymbol{P}^N)$ with respect to Ξ. Namely $\mathrm{Hom}^*(X,\boldsymbol{P}^N)=\coprod_{i\in I} Z_i$ with I being a finite set such that $\Xi_i=\Xi_{(Z_i)}$ is flat for each i with respect to the natural morphism

$$\Xi\subset \boldsymbol{P}^N\times \mathrm{Hom}^*(X,\boldsymbol{P}^N)\to \mathrm{Hom}^*(X,\boldsymbol{P}^N).$$

Every $Y\in \mathscr{F}^a(X)$ is isomorphic to a fibre Ξ_t of Ξ_i over $t\in Z_i$ for some $i\in I$. Set $Z_i^o=\{t\in Z_i; \Xi_t \text{ is smooth}\}$ and $Z_i^a=\{t\in Z_i; \Xi_t\in \mathscr{F}^a(X)\}$. Then $Z_i^a\subset Z_i^o$. By EGA IV [5] (chap. III, p. 183, (iii) of Theorem (12.2.4)), Z_i^o is open in Z_i.

Thus we have a finite number of Noetherian schemes $\{S_j\}$ and smooth morphisms $\Xi_j\to S_j$ such that every $Y\in\mathscr{F}^a(X)$ is isomorphic to a fibre of one of Ξ_i. (For example, set $\{S_j\}=\{Z_i^o\}$.) Moreover, taking a finer stratification, we may assume that all the S_j are smooth.

Lemma 3 (Horikawa [7]. Compare also Fujita [4] p. 789 (4.5) lemma.). *Let X be a complete non-singular variety and let $q:\mathcal{Y}\to S$ be a proper smooth morphism, where S is a regular affine variety. Suppose that we have the following commutative triangle*

$$\begin{array}{ccc} X\times S & \xrightarrow{\ \psi\ } & \mathcal{Y} \\ {\scriptstyle p}\searrow & \circlearrowleft & \swarrow{\scriptstyle q} \\ & S & \end{array}$$

such that ψ is surjective. Then the Kodaira-Spencer map

$$\rho_t : (\Theta_S)_t \to H^1(\mathcal{Y}_t, (\Theta_{\mathcal{Y}/S})_t) \qquad \textit{for all } t \in S$$

is a zero map.

Proof. From the smoothness of p and q, we have a commutative diagram:

$$(*) \qquad \begin{array}{ccc} \Theta_S & \xrightarrow{\text{zero}} & R^1p_*\Theta_{X\times S/S} \\ \| & \circlearrowleft & \downarrow \\ \Theta_S & \longrightarrow & R^1p_*\psi^*\Theta_{\mathcal{Y}/S} \\ \| & \circlearrowleft & \uparrow \\ \rho : \Theta_S & \longrightarrow & R^1q_*\Theta_{\mathcal{Y}/S}. \end{array}$$

The map $\psi_t^1 : H^1(\mathcal{Y}_t, (\Theta_{\mathcal{Y}/S})_t) \to H^1(X, \psi_t^*(\Theta_{\mathcal{Y}/S})_t)$ is injective. Indeed, letting $f : X \to Y$ be a surjective morphism where X and Y are complete non-singular varieties, we shall show that the homomorphism

$$f^p : H^p(Y, \mathcal{F}) \to H^p(X, f^*\mathcal{F})$$

induced from f is injective, where $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-Module of finite type. We often use the following claim:

Let $g : X_1 \to X_2$ and $h : X_2 \to X_3$ be morphisms and let $\mathcal{F}$ be a locally free $\mathcal{O}_{X_3}$-Module of finite type, and if $(h\circ g)^p : H^p(X_3, \mathcal{F}) \to H^p(X_1, (h\circ g)^*\mathcal{F})$ is injective for some $p \geqq 0$, then $h^p : H^p(X_3, \mathcal{F}) \to H^p(X_2, h^*\mathcal{F})$ is also injective. Suppose that $\dim X > \dim Y$. By the Stein-factorization of f, we may assume that fibres of f are connected. In this case, consider a general hyperplane section L of X such that $f|_L : L \to Y$ is surjective. By the claim above, we can replace X by L. Thus it suffices to prove for the case in which $\dim X = \dim Y$. Let $\nu : X_n \to X$ be the normalization. We have the trace homomorphism

$$\mathrm{Tr}_p : H^p(X_n, \nu^* f^*\mathcal{F}) \to H^p(Y, \mathcal{F}).$$

Then $\mathrm{Tr}_p \circ \nu^p \circ f^p$ is a multiple map, which is injective. Hence f^p is injective. Therefore, the injectivity of ψ_t^1 has been proved. By the diagram $(*)$, we have $\rho_t = 0$ for all $t \in S$. Q.E.D.

By virtue of Kodaira-Spencer [1], the smooth family $\mathcal{Y} \xrightarrow{q} S$ is locally trivial if $\dim H^1(\mathcal{Y}_t, (\Theta_{\mathcal{Y}/S})_t)$ is constant and $\rho_t = 0$ for all $t \in S$. Since $\dim H^1(\mathcal{Y}_t, (\Theta_{\mathcal{Y}/S})_t)$ is upper-semi-continuous, we have a stratification $\coprod S_k$ of S such that $\dim H^1(\mathcal{Y}_t, (\Theta_{\mathcal{Y}/S})_t)$ is constant on each stratum S_k and S_k is smooth and connected. Hence, by Lemma 3, the fibre of $\mathcal{Y}_k = q^{-1}(S_k) \to S_k$ are isomorphic to each other.

Applying this result to $\Xi_j \to S_j$, we infer that there are only finitely many manifolds up to isomorphism which appear as a fibre of $\Xi_j \to S_j$ for some j. Therefore $\mathcal{F}^a(X)$ is finite up to isomorphism.

By virtue of a theorem of Kobayashi-Ochiai [9], $\mathcal{F}^a(X)$ is finite up to isomorphism in the strong sense. Hence we complete the proof of Theorem 1. Q.E.D.

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