33. On the Spectra of Laplace Operator on $\Lambda^*(S^n)$

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0. Let Δ be the Laplace operator acting on the space $\Lambda^*(S^n)$ of differential forms on the standard sphere S^n. A. Ikeda and Y. Taniguchi [1] and B. L. Beers and R. S. Millman [2] regarded Δ as the Casimir operator and determined its eigenvalues and multiplicities by using the representation theory.

On the other hand, S. Gallot and D. Meyer [4] tried to determine them by direct computations using harmonic homogeneous forms. But the result for multiplicities contains some errors. In this paper we show the complete result by an elementary method not using the representation theory.

1. Let D be the connection of $\boldsymbol{R}^{n+1}$ and ∇ the connection of S^n induced by the inclusion map ι from S^n into $\boldsymbol{R}^{n+1}$, where we use the canonical metrics. Then, for local vector fields X and Y on S^n, we know

$$D_X Y = \nabla_X Y - \langle X, Y\rangle X_{n+1}$$

and

$$D_X X_{n+1} = X,$$

where X_{n+1} is the locally extended vector field in $\boldsymbol{R}^{n+1}$ from the normal vector of norm 1 at each point of S^n by parallel transportation along the ray issuing from the origin, and X and Y are locally extended in $\boldsymbol{R}^{n+1}$.

Hereafter we extend the local vector field X on S^n so as to satisfy $[X, X_{n+1}]=0$. Also note that $D_{X_{n+1}}X_{n+1}=0$. Now denote by d_0, δ_0 and $\bar{\Delta}$ respectively the differential, its codifferential and Laplace operator on the space $\Lambda^p(\boldsymbol{R}^{n+1})$ of differential p-forms on $\boldsymbol{R}^{n+1}$ associated to D. Then, for any closed p-form α on $\boldsymbol{R}^{n+1}$, we have

$$\begin{aligned}&[\Delta(\iota^*\alpha)-\iota^*(\bar{\Delta}\alpha)]_x(i_1, \cdots, i_p)\\ &\quad = X_{n+1}|_x [X_{n+1}\alpha(i_1, \cdots, i_p)] + (n-2p+2)X_{n+1}|_x\, \alpha(i_1, \cdots, i_p)\end{aligned}$$

at any $x \in S^n$, where $\alpha(i_1, \cdots, i_p)=\alpha(X_{i_1}, \cdots, X_{i_p})$. (Cf. [4].)

Moreover, if α is a harmonic homogeneous p-form of degree k on $\boldsymbol{R}^{n+1}$, we have

$$\Delta(\iota^*\alpha)_x(i_1, \cdots, i_p)=(k+p)(n-p+k+1)\alpha_x(i_1, \cdots, i_p)$$

at any $x \in S^n$. (Cf. [4].)

Let H_k^p be the set of all coclosed harmonic homogeneous p-forms of degree k on $\boldsymbol{R}^{n+1}$ and let V_λ^p denote the subspace of $\Lambda^p(S^n)$ consisting of eigenforms associated to each eigenvalue λ of Δ. Since

$$\iota^* : \sum_{k\geq 0} H_k^p \longrightarrow \Lambda^p(S^n)$$

is injective and its image is dense, we have the isomorphism

$$\iota^* : H_k^p \cap \operatorname{Ker} d_0 \longrightarrow V_{\lambda_k}^p \cap \operatorname{Ker} d,$$

where $\lambda_k=(k+p)(n-p+k+1)$. (Cf. [1], [4].)

Thus, in order to determine the dimension of $V_{\lambda_k}^p \cap \operatorname{Ker} d$, we must calculate the dimension of $H_k^p \cap \operatorname{Ker} d_0$.

2. Let

$$rdr=\sum_{i=1}^{n+1} x^i dx^i \quad \text{and} \quad r\frac{d}{dr}=\sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}.$$

Define a linear operator $e(rdr)$ by

$$e(rdr)\alpha = rdr\wedge\alpha, \qquad \text{for } \alpha\in\Lambda^*(\boldsymbol{R}^{n+1})$$

and denote by $i(r(d/dr))$ the interior product by $r(d/dr)$ on $\Lambda^*(\boldsymbol{R}^{n+1})$. Then the next lemma is obtained by applying H. Cartan's formulas for the Lie derivation to the space P_k^p of homogeneous p-forms of degree k on $\boldsymbol{R}^{n+1}$. (Cf. [1].)

Lemma 1. *For any* $\alpha\in P_k^p$ *we have the following equalities*:

$$(1) \qquad d_0 i\left(r\frac{d}{dr}\right)\alpha + i\left(r\frac{d}{dr}\right)d_0\alpha=(k+p)\alpha,$$

$$(2) \qquad \delta_0 e(rdr)\alpha + e(rdr)\delta_0\alpha = -(n+k-p+1)\alpha.$$

As an easy consequence of this lemma, we have

Corollary 2. P_k^p *decomposes into the direct sum*:

$$(3) \qquad P_k^p=(P_k^p\cap \operatorname{Ker} d_0)\oplus\left(P_k^p\cap \operatorname{Ker} i\left(r\frac{d}{dr}\right)\right) \qquad (k+p\neq 0),$$

$$(4) \qquad P_k^p=(P_k^p\cap \operatorname{Ker} \delta_0)\oplus(P_k^p\cap \operatorname{Ker} e(rdr)) \qquad (n+1-p+k\neq 0).$$

Note that the exceptional cases are $P_0^0=\{\textit{constant functions}\}$ and $P_0^{n+1}=\{a dx^1\wedge\cdots\wedge dx^{n+1};\, a\in\boldsymbol{R}\}$ and both are of dimension 1.

Now we have the direct sum decomposition

$$(5) \qquad P_k^p=r^2P_{k-2}^p\oplus(P_k^p\cap \operatorname{Ker} \bar{\Delta})$$

from the corresponding theorem for functions (cf. [3] or [5]), where we regard that $P_k^p=\{0\}$ for $k<0$.

Furthermore, restricting to $\operatorname{Ker} \delta_0$, we have

Lemma 3. *The following direct sum decomposition holds*:

$$(6) \qquad P_k^p\cap \operatorname{Ker} \delta_0=(r^2P_{k-2}^p\cap \operatorname{Ker} \delta_0)\oplus H_k^p.$$

Proof. It is trivial that $(r^2P_{k-2}^p\cap \operatorname{Ker} \delta_0)\oplus H_k^p\subset P_k^p\cap \operatorname{Ker} \delta_0$. Since $\bar{\Delta}\delta_0=\delta_0\bar{\Delta}$, we have

$$\bar{\Delta} : P_k^p\cap \operatorname{Ker} \delta_0 \longrightarrow P_{k-2}^p\cap \operatorname{Ker} \delta_0$$

and

$$\dim H_k^p+\dim (P_{k-2}^p\cap \operatorname{Ker} \delta_0)\geqq \dim (P_k^p\cap \operatorname{Ker} \delta_0).$$

Now, using (4), we have

$$\begin{aligned}\dim (r^2P_{k-2}^p\cap \operatorname{Ker} \delta_0) &= \dim r^2P_{k-2}^p - \dim (r^2P_{k-2}^p\cap \operatorname{Ker} e(rdr))\\ &= \dim P_{k-2}^p - \dim (P_{k-2}^p\cap \operatorname{Ker} e(rdr))\\ &= \dim (P_{k-2}^p\cap \operatorname{Ker} \delta_0)\end{aligned}$$

and hence

$$\dim H_k^p + \dim(r^2 P_{k-2}^p \cap \operatorname{Ker} \delta_0) \geqq \dim(P_k^p \cap \operatorname{Ker} \delta_0).$$

(Here we note that $n+1-p+k-2 \neq 0$.) Q.E.D.

Remark. From (4), (5) and (6) follows the direct sum decomposition:

$$P_k^p = H_k^p \oplus (r^2 P_{k-2}^p \cap \operatorname{Ker} \delta_0) \oplus (r^2 P_{k-2}^p \cap \operatorname{Ker} e(rdr)) \oplus (P_k^p \cap \operatorname{Ker} \bar{\Delta} \cap \operatorname{Ker} e(rdr)).$$

We may replace $\operatorname{Ker} e(rdr)$ with $\operatorname{Im} e(rdr)$ since $\operatorname{Ker} e(rdr) = \operatorname{Im} e(rdr)$ on P_k^p.

Since the exact sequence

$$0 \longrightarrow \cdots \longrightarrow P_k^p \xrightarrow{\delta_0} P_{k-1}^{p-1} \xrightarrow{\delta_0} P_{k-2}^{p-2} \longrightarrow \cdots \longrightarrow 0$$

follows by (2), we have

$$\dim(P_k^p \cap \operatorname{Ker} \delta_0) = \sum_{i=0}^{\operatorname{Min}(p,k)} (-1)^i \dim P_{k-i}^{p-i}.$$

Next, using the fact that $d_0 \bar{\Delta} = \bar{\Delta} d_0$ and (1), we have the following exact sequence;

$$0 \longrightarrow H_{k+p}^0 \longrightarrow \cdots \longrightarrow H_{k+1}^{p-1} \xrightarrow{d_0} H_k^p \xrightarrow{d_0} H_{k-1}^{p+1} \longrightarrow \cdots \longrightarrow H_0^{p+k} \longrightarrow 0.$$

And hence we have

$$\begin{aligned}
\dim(H_k^p \cap \operatorname{Ker} d_0) &= \sum_{j=1}^{p} (-1)^{j-1} \dim H_{k+j}^{p-j} \\
&= \sum_{j=1}^{p} (-1)^{j-1} \left(\sum_{i=0}^{\operatorname{Min}(p-j,k+j)} (-1)^i (\dim P_{k+j-i}^{p-j-i} - \dim P_{k+j-i-2}^{p-j-i}) \right) \\
&= \sum_{i=1}^{p} (-1)^{i-1} (\dim P_{k+i}^{p-i} - \dim P_{k-i}^{p-i}).
\end{aligned}$$

Since

$$\dim P_k^p = \binom{n+k}{k} \cdot \binom{n+1}{p}$$

where we denote

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!} \qquad \text{for } b>0,$$

$\binom{a}{b}=1$ for $b=0$ and $\binom{a}{b}=0$ for $b<0$, it follows that

$$\dim H_k^p = \sum_{i=0}^{p} (-1)^i \left\{ \binom{n+k-i}{k-i} - \binom{n+k-2-i}{k-2-i} \right\} \cdot \binom{n+1}{p-i}$$

and

$$\dim(H_k^p \cap \operatorname{Ker} d_0) = \sum_{i=1}^{p} (-1)^{i-1} \left\{ \binom{n+k+i}{k+i} - \binom{n+k-i}{k-i} \right\} \cdot \binom{n+1}{p-i}.$$

To simplify these summations, we need the following lemmas which can be proved by induction on the degree p of forms.

Lemma 4. $$\sum_{i=0}^{p} (-1)^i \binom{n+k-i}{k-i} \cdot \binom{n+1}{p-i} = \frac{(n+k+1)!}{p!\,k!\,(n-p)!\,(n+k-p+1)}.$$

Lemma 5. $\sum_{i=1}^{p}(-1)^{i-1}\binom{n+k+i}{k+i}\cdot\binom{n+1}{p-i}$
$$=\frac{(n+k+1)!}{(p-1)!\,k!\,(n-p+1)!\,(p+k)}.$$

By Lemma 4, we have
$$\dim H_k^p=\frac{(n+k-1)!\,(n^3+(3k-p)n^2+(2k^2-(2p-1)k-p-1)n-2pk)}{p!\,k!\,(n-p)!\,(n+k-p+1)(n+k-p-1)}$$
and by Lemmas 4 and 5, we have
$$\dim(H_k^p\cap \operatorname{Ker} d_0)=\frac{(n+k)!\,(n+2k+1)}{(p-1)!\,k!\,(n-p)!\,(n+k-p+1)(k+p)}.$$

Remark. The same calculations using (3) instead of (4) yield $\dim(P_k^p\cap \operatorname{Ker}\bar{\Delta}\cap \operatorname{Ker} d_0)$
$$=\frac{(n+k-1)!\,((p+k-2)n^2+(2k+1)(p+k-2)n+2k(p-1))}{(p-1)!\,k!\,(n-p+1)!\,(p+k)(p+k-2)}.$$

3. Since the direct sum decomposition
$$V_{\lambda_k}^p=(V_{\lambda_k}^p\cap \operatorname{Ker} d)\oplus(V_{\lambda_k}^p\cap \operatorname{Ker}\delta)\qquad(\lambda_k\neq 0)$$
holds and there exists an isomorphism
$$d: V_{\lambda_k}^p\cap \operatorname{Ker}\delta\longrightarrow V_{\lambda_k}^{p+1}\cap \operatorname{Ker} d,$$
we must consider the influence from $(p+1)$-forms.

The eigenvalue ${}^{p+1}\lambda_{k'}=(k'+p+1)(n-p+k')$ of Δ on $\Lambda^{p+1}(S^n)$ is equal to ${}^{p}\lambda_k=(k+p)(n-p+k+1)$ if and only if $n=2p$, and in this case, $k'=k$.

Thus we obtain the following result.

Theorem 6. *The eigenvalue of the Laplace operator on* $\Lambda^p(S^n)$ $(p\neq 0)$ *is of the form*
$$ {}^{p}\lambda_k=(k+p)(n-p+k+1)\qquad k=0,1,2,\cdots,$$
$$ {}^{p+1}\lambda_{k'}=(k'+p+1)(n-p+k')\qquad k'=0,1,2,\cdots.$$
And if $n\neq 2p$, *the multiplicity of* ${}^{p}\lambda_k(\neq 0)$ *is*
$$\frac{(n+k)!\,(n+2k+1)}{(p-1)!\,k!\,(n-p)!\,(n+k-p+1)(k+p)}$$
and the multiplicity of ${}^{p+1}\lambda_k(\neq 0)$ *is*
$$\frac{(n+k)!\,(n+2k+1)}{p!\,k!\,(n-p-1)!\,(n+k-p)(k+p+1)}.$$
If $n=2p$, *the multiplicity of* ${}^{p}\lambda_k={}^{p+1}\lambda_k$ *is*
$$\frac{2(2p+k)!\,(2p+2k+1)}{p!\,(p-1)!\,k!\,(k+p)(k+p)(k+p+1)}.$$

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