31. On the Unique Maximal Idempotent Ideals of Non-Idempotent Multiplication Rings

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In the preceding paper [5], we have defined multiplication rings, shortly M-rings, as rings s.t. for any ideals $\mathfrak{a}, \mathfrak{b}$, with $\mathfrak{a}<\mathfrak{b}$, there exist ideals $\mathfrak{c}, \mathfrak{c}'$, s.t. $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{c}'\mathfrak{b}$; here "$<$" means a proper inclusion. An M-ring is called non-idempotent, if $R>R^2$. We have proved that the unique maximal idempotent ideal $\mathfrak{d}$ of a non-idempotent M-ring can be obtained as an intersection of some ideal sequence $\{\mathfrak{d}_\alpha\}_\wedge$, where $\mathfrak{d}_\alpha$ are defined inductively ([5], Theorem 5): $\mathfrak{d}=\bigcap_{\alpha\in\wedge}\mathfrak{d}_\alpha$. In § 1, we shall prove that $\mathfrak{d}$ is an essential submodule of R, both as a left and also as a right R-module, and at the end of the section we shall give an example of a non-idempotent M-ring with $\mathfrak{d}\neq\{0\}$. If moreover R is left Noetherian, and let N denote the Jacobson radical of R, then by Theorem 5 (i) [5], $N\subseteq\mathfrak{d}$ or $N=\mathfrak{d}_\alpha^j$ for some ordinal α and some positive integer j. If $N=\mathfrak{d}$ or $N=\mathfrak{d}_\alpha^j$, then by Theorem 5 (ii) [5] and Nakayama's lemma $\mathfrak{d}=\{0\}$, so we have to consider the case $N<\mathfrak{d}$ only; so in § 2 we consider left Noetherian non-idempotent M-rings, and prove that any ideal, which is maximal in the set of ideals properly contained in $\mathfrak{d}$, is a prime ideal of R.

1. Non-idempotent M-rings. Lemma 1. *Let R be a non-idempotent M-ring, and let $\mathfrak{a}$ be any ideal, s.t. $\mathfrak{a}\subseteq\mathfrak{d}$ then $\mathfrak{d}\mathfrak{a}=\mathfrak{a}\mathfrak{d}=\mathfrak{a}$; furthermore for an ideal $\mathfrak{d}'$ s.t. $\mathfrak{d}\subseteq\mathfrak{d}'$, $\mathfrak{a}\mathfrak{d}'=\mathfrak{d}'\mathfrak{a}=\mathfrak{a}$.*

Proof. If $\mathfrak{a}=\mathfrak{d}$, there is nothing to prove. If $\mathfrak{a}<\mathfrak{d}$, then $\mathfrak{a}=\mathfrak{d}\mathfrak{b}=\mathfrak{b}'\mathfrak{d}$ for some ideals $\mathfrak{b}, \mathfrak{b}'$, therefore $\mathfrak{a}\mathfrak{d}=\mathfrak{b}'\mathfrak{d}\cdot\mathfrak{d}=\mathfrak{b}'\mathfrak{d}=\mathfrak{a}$. Similarly $\mathfrak{d}\mathfrak{a}=\mathfrak{a}$.

Lemma 2. *Let R be a non-idempotent M-ring, and let $N<\mathfrak{d}$, then $N=\bigcap_{I\in\mathfrak{M}}I=\bigcap_{J\in\mathfrak{N}}J$, where $\mathfrak{M}$ and $\mathfrak{N}$ denote the set of maximal left ideals of R, and all maximal right ideals of R respectively.*

Proof. In general, $NR\subseteq\bigcap_{I\in\mathfrak{M}}I\subseteq N$, and $\bigcap_{I\in\mathfrak{M}}I$ is an ideal of R. By Lemma 1 $N=NR$, hence equality holds.

Theorem 1. *Let R be a non-idempotent M-ring. If $R\neq N$, then $N=\bigcap_{I\in\mathfrak{M}}I=\bigcap_{J\in\mathfrak{N}}J$, where $\mathfrak{M}, \mathfrak{N}$ is the same as Lemma 2.*

Proof. By Proposition 4 [5], $N=R$ or $N\subseteq\mathfrak{d}$. If $N=\mathfrak{d}$, then $\mathfrak{d}=\mathfrak{d}R=NR\subseteq\bigcap_{I\in\mathfrak{M}}I\subseteq\mathfrak{d}$, therefore $\mathfrak{d}=N=\bigcap_{I\in\mathfrak{M}}I$. If $N<\mathfrak{d}$, the results follow by Lemma 2.

Lemma 3. *Let R be a non-idempotent M-ring, and let I be any maximal left ideal of R, then $I\mathfrak{d}=\mathfrak{d}$. The similar results hold for right* *ideals.*

Proof. Assume that $\mathfrak{d} \not\subseteq I$. If $IR \not\subseteq I$, then $(IR, I)=R$ since I is a maximal left ideal of R, therefore $\mathfrak{d}=R\mathfrak{d}=(IR, I)\mathfrak{d}=(I\mathfrak{d}, I\mathfrak{d})=I\mathfrak{d}$, i.e. $I\mathfrak{d}=\mathfrak{d}$. If $IR \subseteq I$, then I is an ideal, hence $I \subseteq \mathfrak{d}$ or $I=\mathfrak{d}_\alpha^\rho$ for some ordinal α and some positive integer ρ, since I is a maximal left ideal, it follows that $I=R^2 \supseteq \mathfrak{d}$, a contradiction. Next let $\mathfrak{d} \subseteq I$, then $\mathfrak{d}=\mathfrak{d}\mathfrak{d} \subseteq I\mathfrak{d}$, i.e. $\mathfrak{d} \subseteq I\mathfrak{d}$, hence $\mathfrak{d}=I\mathfrak{d}$. In either case, we have $\mathfrak{d}=I\mathfrak{d}$.

Theorem 2. *Let R be a non-idempotent M-ring, and let I be any maximal left ideal of R, then for any ideal $\mathfrak{a}$, s.t. $\mathfrak{a} \subseteq \mathfrak{d}$, $I\mathfrak{a}=\mathfrak{a}$. The similar results hold for right ideals.*

Proof. By Lemmas 1 and 3, $I\mathfrak{a}=I\cdot\mathfrak{d}\mathfrak{a}=I\mathfrak{d}\cdot\mathfrak{a}=\mathfrak{d}\mathfrak{a}=\mathfrak{a}$.

Theorem 3. *Let R be a non-idempotent M-ring, and let $\mathfrak{d}\neq\{0\}$, then l-ann $\mathfrak{d}=r$-ann $\mathfrak{d}=\{0\}$ and $\mathfrak{d}\overset{\prime}{\subseteq} R$, i.e. $\mathfrak{d}$ is essential as left R-module and also as a right R-module.*

Proof. Let $\mathfrak{n}$ denote l-ann $\mathfrak{d}=\{x \in R \mid x\mathfrak{d}=\{0\}\}$. If $\mathfrak{n}=\mathfrak{d}_\alpha^j$ for some ordinal α and some positive integer j, $\{0\}=\mathfrak{n}\mathfrak{d}=\mathfrak{d}_\alpha^j\mathfrak{d}=\mathfrak{d}$, a contradiction; if $\mathfrak{n}=\mathfrak{d}$, then $\{0\}=\mathfrak{n}\mathfrak{d}=\mathfrak{d}\mathfrak{d}=\mathfrak{d}$, also a contradiction. If $\mathfrak{n}<\mathfrak{d}$, then by Lemma 1 $\{0\}=\mathfrak{n}\mathfrak{d}=\mathfrak{n}$. Similarly r-ann $\mathfrak{d}=\{0\}$.

Proposition 4. *Let R be a non-idempotent M-ring, and let $\mathfrak{p}$ be any prime ideal of R, s.t. $\mathfrak{p}<\mathfrak{a}$, then $\mathfrak{p}<\mathfrak{a}^n$ for any positive integer n.*

Theorem 4. *Let R be a non-idempotent M-ring, and let $\mathfrak{d}$ be the unique maximal idempotent ideal of R. If $\mathfrak{a}, \mathfrak{b}$ are ideals of R, s.t. $\mathfrak{a}<\mathfrak{b}\subseteq\mathfrak{d}$, then there exist ideals of R $\mathfrak{c}, \mathfrak{c}'$, both contained in $\mathfrak{d}$, s.t. $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{c}'\mathfrak{b}$.*

Proof. It's obvious by Theorem 5 (i) [5] and Lemma 1.

Example. The following is an example of a non-idempotent M-ring with $\mathfrak{d}\neq\{0\}$. Let S be a matrix-ring of a countable degree, generated by countable matrix units $e_{i,j}$ $(i, j=1, 2, \cdots)$ over the rational field. Then S is a simple ring, and $S^2=S$, but does not have an identity. Let $A=pZ_p$, where p is a prime, then ideals of A are p^iZ_p $(i=1, 2, \cdots)$ only. Now we define a ring R as follows: Let $R=(A, S)$, and $(a, s)=(a', s')$, $a, a' \in A$, $s, s' \in S$, if and only if $a=a'$ and $s=s'$; $(a, s)+(a', s')=(a+a', s+s')$, and $(a, s)(a', s')=(aa', as'+sa'+ss')$. Then R is a ring, and $(0, S)$ is an ideal of R, s.t. $(0, S)^2=(0, S)$. Now, let J_i denote p^iZ_p, then (J_i, S) $i\geq 1$ are ideals of R. Let $I=(I_1, I_2)'$ be any ideal of R, where "$'$" means a subdirect sum, and $I_1\subseteq A$, $I_2\subseteq S$, both are projections of I respectively into A and S, and I_1 is an ideal of A, so $I_1=J_i$ for some positive integer i. We assume that $I\neq\{0\}$, then $I\cap S$ is an ideal of S, therefore $I\cap S=S$ or $I\cap S=\{0\}$, since S is a simple ring. In case $I\cap S=S$, $I=(I_1, I_2)'\supseteq(0, S)$, therefore $I_2=S$, hence $I=(J_i, S)'\supseteq(0, S)$ for some integer $i>0$, so $I=(J_i, S)$. In case $I\cap S=\{0\}$, I can not contain any element $(0, x)$, $x\neq 0$. So, let (a_1, s_1) $a_1\neq 0$

be any element of I, then $I \ni (a_1, s_1)(0, t) = (0, a_1t + s_1t)$ for any $t \in S$, therefore $a_1t = -s_1t$ for any $t \in S$. We set $-s_1 = (\beta_{ij})$, and $t = e_{pq}$, then $a_1 = \beta_{pp}$ for any p, a contradiction. Since $R^n = (A^n, S)$, $\mathfrak{d}_1 = \bigcap_{n=1}^{\infty} R^n = \bigcap_{n=1}^{\infty} (A^n, S) = (0, S) \neq 0$, $\mathfrak{d}_1^2 = \mathfrak{d}_1 = \mathfrak{d} \neq \{0\}$.

2. Left Noetherian non-idempotent M-ring. **Proposition 5.** *Let R be a left Noetherian non-idempotent M-ring, and $\mathfrak{p}$ be a prime ideal, s.t. $\mathfrak{p} < N$ then $\mathfrak{p} = \{0\}$.*

Proof. By Proposition 1 [5] $\mathfrak{p} = N\mathfrak{p}$, so by Nakayama's lemma $\mathfrak{p} = \{0\}$.

Theorem 5. *Let R be a left Noetherian non-idempotent M-ring, and also a semi-prime ring. Suppose $N < \mathfrak{d}$, then for any maximal left ideal I of R, l-ann $(I) = \{0\}$, i.e. I is a faithful left R-module.*

Proof. Let I be a maximal left ideal of R, and set $\mathfrak{a} = l\text{-ann}\,(I) = \{x \in R \mid xI = \{0\}\}$. Suppose $\mathfrak{a} \not\subseteq \mathfrak{d}$, then $\mathfrak{a} = \mathfrak{d}_\alpha^\rho$ for some ordinal α and some positive integer ρ, by Theorem 5 (i) [5]. Since R is left Noetherian, $I = Ru_1 + \cdots + Ru_r + Zu_1 + \cdots + Zu_r$, $u_i \neq 0, i = 1, 2, \cdots, r$, where Z denotes the ring of integers. Therefore $\{0\} = \mathfrak{a}I = \mathfrak{a}Ru_1 + \cdots + \mathfrak{a}Ru_r + \mathfrak{a}u_1 + \cdots + \mathfrak{a}u_r = \mathfrak{a}u_1 + \cdots + \mathfrak{a}u_r$, since $\mathfrak{a}R = \mathfrak{d}_\alpha^\rho R = \mathfrak{d}_\alpha^\rho = \mathfrak{a}$. If every u_i, $i = 1, 2, \cdots, r$ belong to R^2, then $I \subseteq R^2$, therefore by the maximality of I $I = R^2$, hence $\{0\} = \mathfrak{a}I = \mathfrak{d}_\alpha^\rho R^2 = \mathfrak{d}_\alpha^\rho = \mathfrak{a}$, i.e. $\mathfrak{a} \subseteq \mathfrak{d}$, a contradiction. Therefore, some $u_i \notin R^2$. Then $\{0\} = \mathfrak{a}u_i = \mathfrak{d}_\alpha^\rho u_i \supseteq \mathfrak{d}u_i$, i.e. $\mathfrak{d}u_i = \{0\}$, by Theorem 3 $u_i = 0$, a contradiction. Thus we conclude that $\mathfrak{a} \subseteq \mathfrak{d}$, hence by Theorem 2 $I\mathfrak{a} = \mathfrak{a}$, therefore $\mathfrak{a}^2 = I\mathfrak{a} \cdot I\mathfrak{a} = I \cdot \mathfrak{a}I \cdot \mathfrak{a} = \{0\}$. Since R is a semi-prime ring, $\mathfrak{a} = \{0\}$.

Proposition 6. *Let R be a left Noetherian non-idempotent M-ring, and let $N < \mathfrak{d}$. Assume that $\mathfrak{a}$ is an ideal of R, properly contained in $\mathfrak{d}$. Let $\mathfrak{p}$ be maximal in the set of ideals of R, s.t. $\mathfrak{a} \subseteq \mathfrak{p} < \mathfrak{d}$, then $\mathfrak{p}$ is a prime ideal of R.*

Proof. We assume that $\mathfrak{p}$ is not a prime ideal of R, then there exist ideals of R $\mathfrak{a}, \mathfrak{b}$, s.t. $\mathfrak{a}\mathfrak{b} \equiv 0$, $\mathfrak{a} \not\equiv 0$, $\mathfrak{b} \not\equiv 0 \pmod{\mathfrak{p}}$. We set $(\mathfrak{a}, \mathfrak{p}) = \mathfrak{a}_1$, $(\mathfrak{b}, \mathfrak{p}) = \mathfrak{b}_1$, then $\mathfrak{a}_1\mathfrak{b}_1 = (\mathfrak{a}, \mathfrak{p})(\mathfrak{b}, \mathfrak{p}) \equiv 0 \pmod{\mathfrak{p}}$, and also $\mathfrak{p} < \mathfrak{a}_1$, $\mathfrak{p} < \mathfrak{b}_1$; of course $\mathfrak{a} \subseteq \mathfrak{a}_1$, therefore by the maximality of $\mathfrak{p}$ $\mathfrak{a}_1 \not\subseteq \mathfrak{d}$, and similarly $\mathfrak{b}_1 \not\subseteq \mathfrak{d}$. Hence by Theorem 5 (i) [5], $\mathfrak{a} = \mathfrak{d}_\alpha^i > \mathfrak{d}$, $\mathfrak{b}_1 = \mathfrak{d}_\beta^j > \mathfrak{d}$ for some ordinals α, β and some positive integers i, j. Therefore $\mathfrak{a}_1\mathfrak{b}_1 \supseteq \mathfrak{d}\mathfrak{d} = \mathfrak{d} > \mathfrak{p}$, hence $\mathfrak{a}_1\mathfrak{b}_1 \not\equiv 0 \pmod{\mathfrak{p}}$, a contradiction. Thus $\mathfrak{p}$ is a prime ideal of R.

Proposition 7. *Under the same assumptions as Proposition 6, let $\mathfrak{p}$ be maximal in the set of ideals of R, s.t. $\mathfrak{a} \subseteq \mathfrak{p} < \mathfrak{d}$, and let I be any left ideal of R s.t. $\mathfrak{p} < I \subseteq \mathfrak{d}$. Then the following statements hold:*

i) $IR = \mathfrak{d}$

ii) $I^2 = \mathfrak{d}I$

iii) $\mathfrak{d} = I\mathfrak{d}$

iv) $I\mathfrak{p} = \mathfrak{p}$

v) I^2 is an idempotent left ideal of R

vi) $\mathfrak{p}\subseteq I^n$ for any positive integer n.

Proof. i) Since $\mathfrak{p}<I\subseteq\mathfrak{d}$, $\mathfrak{p}=\mathfrak{p}R\subseteq IR\subseteq\mathfrak{d}R=\mathfrak{d}$. By the maximality of $\mathfrak{p}$, $\mathfrak{p}=IR$ or $IR=\mathfrak{d}$. But the former does not occur.

ii) Using i) $I^2\supseteq I\cdot RI=IR\cdot I=\mathfrak{d}I\supseteq I^2$, therefore $I^2=\mathfrak{d}I$.

iii) Using the results i), ii), $I\mathfrak{d}=I\cdot IR=I^2\cdot R=\mathfrak{d}I\cdot R=\mathfrak{d}\cdot IR=\mathfrak{d}\mathfrak{d}=\mathfrak{d}$.

iv) By iii) $\mathfrak{p}\supseteq I\mathfrak{p}=I\cdot\mathfrak{d}\mathfrak{p}=I\mathfrak{d}\cdot\mathfrak{p}=\mathfrak{d}\mathfrak{p}=\mathfrak{p}$, therefore $I\mathfrak{p}=\mathfrak{p}$.

v) By the results ii), iii), $I^3=I\cdot I^2=I\cdot\mathfrak{d}I=I\mathfrak{d}\cdot I=\mathfrak{d}I=I^2$, i.e. $I^3=I^2$, therefore I^2 is an idempotent left ideal of R.

vi) By iv) $\mathfrak{p}=I\mathfrak{p}\subseteq I\cdot I=I^2$, so $\mathfrak{p}\subseteq I^n$ for any positive integer n.

References

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