30. *A Theorem of Helson and Sarason in Uniform Algebras*

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1. In this note we shall give an extension of a theorem of Helson and Sarason in their paper "Past and Future" to the case of uniform algebra with a unique representing measure. Let X be a compact Hausdorff space and A be a uniform algebra on X, that is a uniformly closed, point separating algebra of continuous, complex valued functions on X containing the constants. Assume, always from now on, that m is a complex homomorphism of A such that m has a unique representing measure dm on X. Define H^p as the closure of A in $L^p(dm)$ (norm closure for $1\leqq p<\infty$; weak*-closure for $p=\infty$). We put $A_0=\left\{f\in A \,\middle|\, \int f dm=0\right\}$ and $H_0^p=\left\{f\in H^p \,\middle|\, \int f dm=0\right\}(1\leqq p\leqq\infty)$. We denote by A_0^n (resp. $(H_0^\infty)^n$) the ideal generated by products of n elements in A_0 (resp. H_0^∞). H_0^∞ is said to be simply invariant if $[A_0H_0^\infty]_*\subsetneqq H_0^\infty$, where $[B]_*$ denotes the weak*-closure of B. Let $G(m)$ be the Gleason part of m, that is, $G(m)$ is the set of all complex homomorphisms σ of A such that the operator norm of $\sigma-m$ is strictly smaller than 2. In our situation the following conditions are equivalent:

(i) H_0^∞ is simply invariant.

(ii) There exists an inner function Z such that $H_0^\infty=ZH^\infty$ (this function is determined uniquely up to multiplication of constants of modulus 1 and is called "Wermer's embedding function").

(iii) $G(m)\neq\{m\}$.

Now let ν be a positive finite Baire measure on X. We denote $\rho_n(\nu)$ by

$$\rho_n(\nu)=\sup\left|\int fg d\nu\right|$$

where f and g range over the elements of A and A_0^n, respectively, subject to the restriction

$$\int |f|^2 d\nu\leqq 1 \quad\text{and}\quad \int |g|^2 d\nu\leqq 1.$$

Clearly $1\geqq\rho_1(\nu)\geqq\rho_2(\nu)\geqq\cdots\geqq 0$.

Our result is the following

Theorem. (A) *In the case* $G(m)=\{m\}$, $\lim_{n\to\infty}\rho_n(\nu)=0$ *if and only*

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if $d\nu = c\,dm$ for some constant c.

(B) *In the case $G(m) \neq \{m\}$, $\lim_{n\to\infty} \rho_n(\nu) = 0$ if and only if $d\nu$ is of the form*

$$d\nu = |P(Z)|^2 \exp(u(Z) + \tilde{v}(Z))dm,$$

where Z is the Wermer's embedding function, P is an analytic polynomial, u and v are real valued continuous functions on the unit circle T and $\tilde{v}$ is the usual conjugate function of v.

We note first that using Forelli's lemma ([2], Lemma 7.3) one can easily show that if $\rho_n(\nu) < 1$, $d\nu$ is absolutely continuous with respect to dm, i.e., $d\nu = w\,dm$ for some $0 \leqq w \in L^1(dm)$. Furthermore, by Proposition 2 in Ohno [5], we have $\log w \in L^1(dm)$ and so $w > 0$ m-a.e. Hence, to prove Theorem we may assume that $d\nu$ has the above form.

Proof of (A). "If" part is clear. Next assume $\lim_{n\to\infty} \rho_n(\nu) = 0$. Since $w > 0$, it is easily shown that $H_0^\infty = [A_0^n]_*$ and $H_0^\infty \subset [A_0^n]_{wdm}$, $n = 1, 2, \cdots$, where $[E]_{wdm}$ denotes the $L^2(wdm)$-closure of E. Suppose w is not a constant. Then, since $A + \overline{A}_0$ is weak*-dense in $L^\infty(dm)$, there exists an $f \in A_0$ with $\int f w\,dm \neq 0$. However $f \in A_0 \subset [A_0^n]_{wdm}$. Hence we have

$$\rho_n(\nu) \geqq \left| \int 1 f w\,dm \right| \Big/ \left(\int w\,dm \int |f|^2 w\,dm \right)^{1/2} > 0,$$

which is a contradiction.

Proof of (B). By virtue of the following Lemmas 1, 2 and 3 one can prove (B) by translating the proofs of Theorems 3, 4, 5 in Helson-Sarason [3] and Theorem A in Sarason [7] word for word to our case, replacing the function $\chi = e^{i\theta}$ and the space $C(T)$ of all continuous functions on the unit circle T by our Wermer's embedding function Z and $C(Z) = \{f(Z) \mid f \in C(T)\}$, respectively.

Lemma 1. *For every Lebesgue measurable set E on the unit circle T we have $L(E) = m\{x\,;\, Z(x) \in E\}$, where L is the normalized Lebesgue measure on T ([10], Corollary 1).*

Lemma 2. *Let k be a positive integer. If $Z^k s \in H^{1/2}$ and $S \geqq 0$, then S has the form*

$$S = |P(Z)|^2,$$

where P is an analytic polynomial on T of degree k.

Proof. Put $g = S + 1$. Then $Z^k g \in H^{1/2}$ and

$$\infty > \int \log |Z^k g|\,dm = \int \log |g|\,dm \geqq \int \log 1\,dm > -\infty.$$

By Theorem 2 in Gamelin [1], it follows that there exist an inner function q in H^∞ and an outer function P_1 in H^2 such that $Z^k g = qP_1^2$. By the same argument as in the proof of Theorem 6 in [5] we see that P_1 has the form

$$P_1 = b_0 + b_1 Z + \cdots + b_k Z^k.$$

Put

$$R(e^{i\theta})=|b_0+b_1e^{i\theta}+\cdots+b_ke^{ik\theta}|^2-1.$$

Since, $R(Z)=|P_1|^2-1\geqq 0$ m-a.e., we have by Lemma 1 $R(e^{i\theta})\geqq 0$ a.e. on T. Since $R(e^{i\theta})$ is a non-negative trigonometric polynomial of degree k, it has a representation $R(e^{i\theta})=|P(e^{i\theta})|^2$ with an analytic polynomial P of degree k. It follows that

$$S=g-1=|P_1|^2-1=R(Z)=|P(Z)|^2.$$

Remark. This is an extension of abstract Neuwirth and Newmann's theorem in Yabuta [9].

Lemma 3. *$H^\infty+C(Z)$ is closed in $L^\infty(dm)$.*

One can prove this in a quite similar way to the proof of Theorems 1, 2 in Rudin [6].

2. Related results. By similar arguments one obtains the following propositions.

Proposition 1. *Let $G(m)\neq\{m\}$. Then $\rho_n(\nu)=0$ if and only if $d\nu$ has the form*

$$d\nu=|P(Z)|^2\,dm,$$

where P is an analytic polynomial on T of degree less than n.

Proposition 2. *Let $G(m)=\{m\}$. Then $\rho_n(\nu)<1$ if and only if $d\nu$ has the form*

$$d\nu=\exp\,(r+\tilde{s})dm,$$

where r is a real valued bounded function and $\tilde{s}$ is the conjugate function of a real valued function s with bound strictly smaller than $\pi/2$.

Remark. Proposition 2 for the case $G(m)\neq\{m\}$ is treated in Ohno [5].

References

[1] T. W. Gamelin: H^p spaces and extremal functions in H^1. Trans. Amer. Math. Soc., **124**, 158–167 (1966).

[2] ——: Uniform Algebras. Prentice-Hall, Englewood Cliffs, New Jersey (1969).

[3] H. Helson and D. Sarason: Past and Future. Math. Scand., **21**, 5–16 (1967).

[4] Y. Ohno: Remarks on Helson-Szegö problems. Tôhoku Math. J., **18**, 54–59 (1966).

[5] ——: Helson-Szegö-Sarason theorem for Dirichlet algebras. Tôhoku Math. J., **31**, 71–79 (1979).

[6] W. Rudin: Spaces of type $H^\infty+C$. Ann. Inst. Fourier, Grenoble, **25**, 99–125 (1975).

[7] D. Sarason: An addendum to "Past and Future". Math. Scan., **30**, 62–64 (1972).

[8] T. P. Srinivasan and J. Wang: Weak*-Dirichlet algebras. Function Algebras (Tulane, 1965), Scott Foresman, Chicago, pp. 216–249 (1966).

[9] K. Yabuta: A note on extremum problems in abstract Hardy spaces. Arch. Math., **23**, 54–57 (1972).

[10] K. Yabuta: On bounded functions in the abstract Hardy space theory. II. Tôhoku Math. J., **26**, 513–533 (1974).