28. *The Eigenvalues of the Laplacian and Perturbation of Boundary Condition*

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§. Introduction. Let Ω be a bounded domain in $\mathbf{R}^n$ with C^∞ boundary γ. Consider eigenvalue problem for the Laplacian with the third boundary condition;

$$(1) \qquad (-\Delta-\lambda)u(x)=0, \qquad x\in\Omega,$$

$$(2)_\rho \qquad \frac{\partial u}{\partial \nu_x}(x)+\rho(x)u(x)=0, \qquad x\in\gamma,$$

where ν_x denotes the exterior unit normal vector at x, and $\rho(x)$ is a function in the Hölder space $C^{2+\theta}(\gamma)$ $(0<\theta<1)$.

Let $\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots$ be the eigenvalues of the problem (1), $(2)_\rho$, then these are functionals of ρ.

Our main result is

Theorem 1. *There is a residual subset B of the Banach space $C^{2+\theta}(\gamma)$ such that for any $\rho\in B$ all the eigenspaces of the problem* (1), $(2)_\rho$ *are of dimension one.*

We call a subset B in $C^{2+\theta}(\gamma)$ residual if it is a countable intersection of open dense subsets of $C^{2+\theta}(\gamma)$.

In Fujiwara-Tanikawa-Yukita [2], they studied the eigenvalue problem of the Laplacian with Dirichlet condition at the boundary γ. Their result is as follows;

(A) If the boundary γ of domain is in residual subset of the Hilbert manifold of the totality of the boundary, then all eigenvalues are simple eigenvalue.

Their proof heavily depends on the abstract transversality theorem of Banach manifold given by Uhlenbeck [3]. Also they used Hadamard's variational formula and showed that the theorem of Uhlenbeck is applicable to their proof of (A).

In our case too, we shall use a variational formula of $(-\Delta+M)^{-1}$, M being large, under the perturbation of $\rho(x)$. This will be proved in Theorem 2.

§ 1. Variational formula of the Green kernel under the perturbation of boundary condition. Let m be a fixed number satisfying $m<\lambda_1$, where λ_1 is the smallest eigenvalue of $-\Delta$ with the boundary condition $(2)_\rho$. Let $G_\rho(x,y)$ be the Green kernel of $-\Delta-m$ with the condition $(2)_\rho$. We fix $\kappa(x)\in C^{2+\theta}(\gamma)$. Then we have the following

Theorem 2. *For fixed $x, y \in \Omega$ snch that $x \neq y$, we have*

$$\delta G_\rho(x, y)=\int_\gamma G_\rho(x, z)G_\rho(y, z)\kappa(z)d\sigma_z,$$

where we put

$$\delta G_\rho(x, y)=\lim_{\varepsilon\to 0}\varepsilon^{-1}(G_{\rho+\varepsilon\kappa}(x, y)-G_\rho(x, y)),$$

and $d\sigma_z$ denotes the surface element of γ.

Proof. We have

$$\begin{aligned}&\varepsilon^{-1}(G_{\rho+\varepsilon\kappa}(x, y)-G_\rho(x, y))\\ &\quad=\varepsilon^{-1}\int_\Omega \{G_{\rho+\varepsilon\kappa}(x, z)(-\Delta_z-m)G_\rho(y, z)\\ &\qquad-G_\rho(y, z)(-\Delta_z-m)G_{\rho+\varepsilon\kappa}(x, z)\}dz\\ &\quad=\int_\gamma G_{\rho+\varepsilon\kappa}(x, z)G_\rho(y, z)\kappa(z)d\sigma_z.\end{aligned}$$

For fixed $x \in \Omega$, we can prove that $G_{\rho+\varepsilon\kappa}(x, z)-G_\rho(x, z)$ converges to zero uniformly in $z \in \bar{\Omega}$. The proof is the same as in Fujiwara-Ozawa [1].

§ 2. Proof of Theorem 1. Let

$$\Sigma=C^{2+\theta}(\gamma),$$

and

$$\Pi=\left\{u \in L^2(\Omega)\,;\int_\Omega |u(x)|^2\,dx=1,\ u: \textit{real}\right\},$$

then Π forms a separable Hilbert manifold. The eigenvalue problem (1), $(2)_\rho$ is equivalent to the equation $(I-(\lambda-m)G_\rho)u(x)=0$, where G_ρ denotes the green operator defined by

$$G_\rho u(x)=\int_\Omega G_\rho(x, y)u(y)dy.$$

We define the following map:

$$\begin{aligned}&\Phi: \Sigma\times\Pi\times\mathrm{R}\mapsto L^2(\Omega)\\ &\qquad\qquad\cup\!\!\!\!\!\;\;\;\\ &(\rho, u, \lambda)\mapsto\Phi(\rho, u, \lambda)=(I-(\lambda-m)G_\rho)u.\end{aligned}$$

Let Φ_ρ denote the map defined by

$$\begin{aligned}&\Phi_\rho: \Pi\times\mathrm{R}\to L^2(\Omega)\\ &(u, \lambda)\to\Phi_\rho(u, \lambda)=\Phi(\rho, u, \lambda).\end{aligned}$$

Then we have

Lemma 1. *The mapping Φ_ρ is a smooth Fredholm mapping of index* 0.

Lemma 2. *The following two statements are equivalent:*

(2.2) 0 *is a regular value of Φ_ρ.*

(2.3) *The eigenvalue of the equation* (1), $(2)_\rho$ *are all simple.*

We omit the proof of Lemmas 1 and 2. See Uhlenbeck [3]. Next we have

Proposition 1. $0 \in L^2(\Omega)$ *is a regular value of the mapping Φ.*

Proof. Assume $(\rho, u, \lambda) \in \Phi^{-1}(0)$. We have to show that the image of $\delta\Phi(\rho, u, \lambda)$ coincides with $L^2(\Omega)$, where $\delta\Phi(\rho, u, \lambda)$ is the differential of Φ at (ρ, u, λ). We have

$$(\delta\Phi(\rho, u, \lambda))(\delta\rho, \delta u, \delta\lambda)=\delta\lambda\cdot G_\rho u+(I-(\lambda-m)G_\rho)\delta u+(\lambda-m)(\delta G_\rho)u.$$

The condition that δu lies in the tangent space of Π at u is equivalent to

$$\int_\Omega \delta u(x)u(x)dx=0.$$

We assume that v is orthogonal to the image of $\delta\Phi(\rho, u, \lambda)$. Then we have

$$\begin{aligned}(2.4)\quad \delta\lambda\int_\Omega (G_\rho u)(x)v(x)dx&+\int_\Omega (I-(\lambda-m)G_\rho)\delta u(x)v(x)dx\\ &+(\lambda-m)\cdot\int_\Omega (\delta G_\rho)u(x)v(x)dx=0,\end{aligned}$$

for any $(\delta\rho, \delta u, \delta\lambda)\in T_{(\rho,u,\lambda)}$, where $T_{(\rho,u,\lambda)}$ denotes the tangent space to $\Sigma\times\Pi\times\mathrm{R}$ at (ρ, u, λ). By the choice of m, we have $\lambda-m\neq 0$, then (2.4) is equivalent to the following system of equations:

$$(2.5)\quad \int_\Omega (G_\rho u)(x)v(x)dx=0,$$

$$(2.6)\quad \int_\Omega (I-(\lambda-m)G_\rho)\delta u(x)v(x)dx=0,$$

$$(2.7)\quad \int_\Omega (\delta G_\rho)u(x)v(x)dx=0.$$

Since $(I-(\lambda-m)G_\rho)$ is a bounded symmetric operator in $L^2(\Omega)$, (2.6) implies that $(I-(\lambda-m)G_\rho)v(x)=0$. Then by the regularity theorem of elliptic boundary value problem, we have $v(x)\in C^{2+\theta}(\bar{\Omega})$. We know by (2.7) and Theorem 2 that

$$\left(\int_\Omega G_\rho(x, z)v(x)dx\right)\left(\int_\Omega G_\rho(y, z)u(y)dy\right)=0,$$

for every $z\in\gamma$. Since $u(x), v(x)\in \operatorname{Ker}(I-(\lambda-m)G_\rho)$, the above equation turns out to be $u(z)v(z)\equiv 0$ on γ. Hence there is an open subset ω of γ such that $u(z)=0$ or $v(z)=0$ on ω holds. Assume that $u(z)=0$ on ω, then $\dfrac{\partial u}{\partial\nu_z}(z)=0$ on ω because of $(2)_\rho$. By the uniqueness theorem of Holmgren and the real analyticity of the solution in the interior of Ω, we obtain $u(x)\equiv 0$ in Ω which contradicts the fact $u\in\Pi$. Thus $v\equiv 0$ on ω. Just as above discussions, we can prove $v\equiv 0$ in Ω. Hence the image of $\delta\Phi(\rho, u, \lambda)$ is dense in $L^2(\Omega)$. On the other hand, the image of $(I-(\lambda-m)G_\rho)$ is closed and has finite codimension in $L^2(\Omega)$. Therefore Proposition 1 is proved.

By Lemmas 1, 2, Proposition 1 and Uhlenbeck's transversality theorem in [3], we get Theorem 1.

References

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[3] Uhlenbeck, K.: Generic properties of eigen functions. Amer. J. Math., **98**, 1059–1078 (1976).