1. *A Simplified Derivation of Mikusiński's Operational Calculus*

By Shûichi OKAMOTO
Department of Mathematics, Gakushuin University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 16, 1979)

§ 1. **Introduction.** O. Heaviside introduced in 1899 an operational calculus [1] which he successfully applied to the integration of linear ordinary differential equations with constant coefficients. In his calculus occured certain operators whose interpretation as given by Heaviside and his successors is not only difficult to justify but also the range of validity of this calculus developed so far remains unclear. In 1949, J. Mikusiński inaugurated the theory [2] of convolution quotients by which he provided a clear and simple operational calculus well-suited for the purpose. His theory is based upon Titchmarsh's theorem concerning the vanishing of convolution of two continuous functions defined on $[0, \infty)$.

The purpose of the present paper is to show that we are able to simplify Mikusiński's operational calculus in such a way that we need not appeal to Titchmarsh's theorem at all. It is to be noted that the author has given in Okamoto [5] another approach which does not appeal to Titchmarsh's theorem.

§ 2. **The convolution ring $\mathfrak{C}$ and the ring $\mathfrak{C}_H$.** We denote by $\mathfrak{C}$ the totality of the complex number valued continuous functions f defined on $[0, \infty)$. In this paper, we write such functions by $\{f(t)\}$ or simply by f, while $f(t)$ means the value at t of the function f.

For $f, g \in \mathfrak{C}$, let the addition of two functions f and g be defined by

$$f+g=\{f(t)+g(t)\} \tag{1}$$

and the multiplication of f and g by the convolution:

$$fg=\left\{\int_0^t f(t-u)g(u)du\right\}. \tag{2}$$

Then we see that $\mathfrak{C}$ is a commutative ring with respect to this addition and this multiplication. We call it the *convolution ring*.

Throughout this paper, h will denote the *constant function* $\{1\}$. So we get

$$h^n=\left\{\frac{t^{n-1}}{(n-1)!}\right\} \qquad n=1,2,3,\cdots. \tag{3}$$

Furthermore, for any $f \in \mathfrak{C}$, we have

$$hf=\left\{\int_0^t f(u)du\right\}$$

so that h may be regarded as the *operator of integration.* Hence the equality $hf=0$ $(=\{0\})$ implies that $f=0$. Thus, by putting $H=\{k; k=h^n, n=1,2,3,\cdots\}$, we have the following

Proposition 1. *For $k\in H$ and $f\in\mathfrak{C}$, the equality $kf=0$ implies that $f=0$.*

Therefore we can construct the ring $\mathfrak{C}_H$ of fractions of $\mathfrak{C}$ with denominators belonging to H:

$$\mathfrak{C}_H=\left\{\frac{f}{k}; f\in\mathfrak{C} \text{ and } k\in H\right\}$$

where the equality is defined by

$$\frac{f}{k}=\frac{f'}{k'}\Longleftrightarrow k'f=kf' \tag{4}$$

and the addition and the multiplication are defined by the usual rules about fractions.

By the mapping from $\mathfrak{C}$ to $\mathfrak{C}_H$

$$f\longmapsto\frac{fk}{k} \tag{5}$$

the ring $\mathfrak{C}$ can be isomorphically imbedded into the ring $\mathfrak{C}_H$ so that we shall identify f with fk/k. For any complex number α, we write $[\alpha]$ for the element $\{\alpha\}/h$ in $\mathfrak{C}_H$. Then we have

$$[\alpha]+[\beta]=[\alpha+\beta], \qquad [\alpha][\beta]=[\alpha\beta] \tag{6}$$

and

$$[\alpha]f=\alpha f \ (=\{\alpha f(t)\}), \qquad [\alpha]\frac{f}{k}=\frac{\alpha f}{k}\ \left(=\frac{\{\alpha f(t)\}}{k}\right) \tag{7}$$

where $\alpha,\beta\in \boldsymbol{C}$, $f\in\mathfrak{C}$ and $k\in H$. Thus, by (6), $[\alpha]$ can be identified with the complex number α, and by (7) we see that the effect of the multiplication by $[\alpha]$ is exactly the *α-times multiplication.* In particular, [1] is the *unit element* of $\mathfrak{C}_H$ which will be written by 1.

Proposition 2. *For $\alpha_i\in\boldsymbol{C}$, $i=1,2,\cdots,n$ we have*

$$\begin{aligned}(n!)^{-1}\{1\}^{n-1}\{t^n+\alpha_1t^{n-1}+\cdots+\alpha_n\}\\ =\{t-\beta_1\}\cdot\{t-\beta_2\}\cdots\{t-\beta_n\}\end{aligned} \tag{8}$$

where β_i are roots of the polynomial

$$t^n+\frac{\alpha_1}{n}t^{n-1}+\frac{\alpha_2}{n(n-1)}t^{n-2}+\cdots+\frac{\alpha_n}{n!}.$$

Proof. $\{t^n+\alpha_1t^{n-1}+\cdots+\alpha_n\}$

$$\begin{aligned}
&=n!\,h^{n+1}+(n-1)!\,\alpha_1h^n+\cdots+\alpha_nh && \text{(by (3))}\\
&=n!\,h\left(h^n+\left[\frac{\alpha_1}{n}\right]h^{n-1}+\cdots+\left[\frac{\alpha_n}{n!}\right]\right) && \text{(by (7))}\\
&=n!\,h(h-[\beta_1])(h-[\beta_2])\cdots(h-[\beta_n]) && \text{(by (6))}\\
&=n!\,h^{1-n}(h^2-\{\beta_1\})(h^2-\{\beta_2\})\cdots(h^2-\{\beta_n\})\\
&=n!\,h^{1-n}\{t-\beta_1\}\{t-\beta_2\}\cdots\{t-\beta_n\}.
\end{aligned}$$

Hence, by multiplying both sides by $(n!)^{-1}h^{n-1}$, we obtain (8).

§3. The ring of convolution quotients. Let α be a complex number and f be a continuous function $\in \mathfrak{C}$. Assume that $pf=0$ with $p=\{t-\alpha\}$. Then, by (2), we see that f must satisfy the differential equation

$$\alpha f' - f = 0 \qquad f(0)=0.$$

Hence we must have $f=0$. Therefore, by virtue of Proposition 2 we see that, even *if p is a non-zero general polynomial in t, the equality implies that $f=0$.*

Let now P denote the totality of non-zero polynomials in t. Then, as in the case of the ring $\mathfrak{C}_H$, we can construct the ring $\mathfrak{C}_P$ of fractions of $\mathfrak{C}$ with denominators belonging to P. Since the set H is contained in P, we can consider the ring $\mathfrak{C}_H$ as a subring of $\mathfrak{C}_P$. In particular, any element f of $\mathfrak{C}$ may be identified with the element $fk/k=fp/p$ of $\mathfrak{C}_P$ where $k \in H$ and $p \in P$. The ring $\mathfrak{C}_P$ will be called the *ring*[1] *of convolution quotients.* Henceforth we shall call these convolution quotients $\in \mathfrak{C}_P$ as *operators.*

The differential operator s. We define the operator s by

$$s=\frac{1}{h}=\frac{h}{h^2} \tag{9}$$

and call it the *differential operator*, since we have

Proposition 3. *If $f \in \mathfrak{C}$ has a continuous derivative f', then*

$$f' = sf - f(0) \tag{10}$$

where $f(0)=\{f(0)\}/h=[f(0)]$.

Proof. We have

$$hf' = \left\{\int_0^t f'(u)du\right\} = \{f(t)-f(0)\} = f - [f(0)]h.$$

Hence we obtain (10).

Corollary. *For n-times continuously differentiable function $f \in \mathfrak{C}$, we have*

$$f^{(n)} = s^n f - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \tag{11}$$

where $f^{(j)}(0)=\{f^{(j)}(0)\}/h=[f^{(j)}(0)]$ $(j=1,2,3,\cdots,n)$.

Rational functions of s. A non-zero polynomial in s is a rational function of h so that we can represent it as a fraction whose denominator and numerator both belong to P. Hence its inverse exists in $\mathfrak{C}_P$. Therefore we can manipulate rational functions of s in $\mathfrak{C}_P$. In particular, we have the following

Proposition 4. *The following identity holds good:*

$$\frac{1}{s-\alpha} = \{e^{\alpha t}\}, \tag{12}$$

where in the left side term α should be understood as $[\alpha]$.

Proof. We have

1) We might express as "the (special) ring" to distinguish it from the ring $\mathfrak{C}_H$ and Mikusiński's field of convolution quotients.

$$\frac{1}{s-\alpha}=\frac{1}{1/h-\{\alpha\}/h}=\frac{h^2}{h-\{\alpha\}h}=\frac{\{t\}}{\{1-\alpha t\}}.$$

It is trivial that $\{1-\alpha t\}\{e^{\alpha t}\}=\{t\}$. Thus we have proved (12).

Corollary.

$$\frac{1}{(s-\alpha)^n}=\left\{\frac{t^{n-1}}{(n-1)!}e^{\alpha t}\right\}. \tag{13}$$

Proof. For $n=2$, we have

$$\frac{1}{(s-\alpha)^n}=\{e^{\alpha t}\}\{e^{\alpha t}\}=\left\{\frac{t}{1!}e^{\alpha t}\right\}.$$

The general case may be proved by induction.

Remark. By (6) and (7), we may manipulate rational functions of s in the same way as ordinary rational functions. Thus we can decompose such rational functions of s in partial fractions.

§ 4. Applications to differential equations and integral equations. We now have all the tools required for solving[2] a linear ordinary differential equation with constant coefficients

$$f^{(n)}+\alpha_1 f^{(n-1)}+\cdots+\alpha_n f=g\in C$$

and also a Volterra integral equation of such a type that in its integrated part $\{K(t)\}\cdot\{f(t)\}$ the kernel $\{K(t)\}$ is given by a linear combination of $\{t^n e^{\alpha t}\}$'s. In the case of the above differential equation, we rewrite, by (11), the equation in the operator form. Then we solve it with respect to the unknown function f obtaining

$$f=\frac{p_1(s)+g}{p_2(s)},$$

where $p_1(s)$ and $p_2(s)$ are polynomials in s such that $p_2(s)\not\equiv 0$ and the degree of $p_1(s)$ is smaller than that of $p_2(s)$. Hence, by (13), we get the explicit expression of the solution $\{f(t)\}\in C$.

In the case of integral equation, we can rewrite, by (13), the equation in the operator form, and so we obtain the solution similarly as above.

Remark. By introducing the notion of the convergence[3] in the same way as in Mikusiński's, we also have some other applications of our calculus.

References

[1] O. Heaviside: Electromagnetic Theory. London (1899).
[2] J. Mikusiński: Sur les fondaments du calcul opératoire. Studia Math., **11**, 41–70 (1949).
[3] ——: Operational Calculus. Pergamon Press (1959).
[4] A. Erdélyi: Operational Calculus and Generalized Functions. Holt, Rinehart and Winston (1962).

2) Cf. examples of those given in [3], [4] and [6].
3) See Mikusiński [3] or Erdélyi [4].

[5] S. Okamoto: A difference approach to Mikusiński's operational calculus. Proc. Japan Acad., **54**A, 303–308 (1978).
[6] K. Yosida: Functional Analysis. 5th ed., Springer-Verlag (1978).