18. *Fundamental Solution of Partial Differential Operator of Schrödinger's Type. III*

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§1. **Introduction.** The aim of this note is to construct, following Feynman's idea, the fundamental solution of the initial value problem for time dependent Schrödinger equation

$$\frac{1}{\lambda}\frac{\partial}{\partial t}u(t,x)+\frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{\lambda}\frac{\partial}{\partial x_j}\right)^2u(t,x)+V(t,x)u(t,x)=0 \tag{1}$$

for $(t,x)\in \boldsymbol{R}\times\boldsymbol{R}^n$ and

$$u(s,x)=\varphi(x). \tag{2}$$

Here $\lambda=i\hbar^{-1}$ is a purely imaginary parameter and $\hbar$ is a small parameter $0<\hbar\ll 1$. The potential $V(t,x)$ is assumed to satisfy the following two conditions.

(V-I) $V(t,x)$ is a real valued function which is continuous in $t\in\boldsymbol{R}$ and infinitely differentiable in $x\in\boldsymbol{R}^n$.

(V-II) For any multi-index α with length $|\alpha|\geq 2$, there is a continuous positive function $C_\alpha(t)$ of t such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(t,x)\right|\leq C_\alpha(t) \tag{3}$$

for any $x\in\boldsymbol{R}^n$.

Feynman's path integral has been discussed by many authors. See, for examples, Feynman [3], Nelson [8], Itô [6], Fujiwara [4], Albeverio-Krohn [1], Keller-McLaughlin [7] and their references. Since we use L^2-theory of oscillatory integral transforms, discussions of the present note seem to contain new results. In particular, we can prove that the sequence of operators which approximates the Feynman path integral converges not merely in strong topology, but also in the uniform operator topology. (In this respect, see [7].)

Assumptions (V-I) and (V-II) are rather severe. However, following examples satisfy them.

1) $V(t,x)=\sum_{|\alpha|\leq 2}a_\alpha(t)x^\alpha$, $a_\alpha(t)$ being real and continuous function of t. If, in particular, $V(t,x)=|x|^2$, this is the potential of a Harmonic oscillator. Since positivity of $V(t,x)$ is not assumed, $V(t,x)=-|x|^2$ also satisfies (V-I) and (V-II).

2) $V(t,x)$ is a smooth potential of long range.

3) $V(t,x)$ is a smooth oscillatory or a periodic potential.

§ 2. **Parametrix.** The classical mechanics corresponding to (1) is described by Lagrangean function

$$L(t, x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 - V(t, x) \tag{4}$$

and Euler's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{5}$$

with initial value

$$x|_{t=s} = y \qquad \dot{x}|_{t=s} = \eta. \tag{6}$$

Let $x(t, s, y, \eta)$ denote the solution of this equation.

Let T be an arbitrarily fixed positive number. We can easily prove the following

Proposition 1. *Assume that $V(t,x)$ satisfies (V-I) and (V-II). Then there exists a positive constant $\delta_1(T)$ such that the mapping*

$$\boldsymbol{R}^n \ni \eta \to x = x(t, s, y, \eta) \in \boldsymbol{R}^n \tag{7}$$

is a global diffeomorphisms of $\boldsymbol{R}^n$ provided that $|t| \leq T$, $|s| \leq T$ and $|t-s| \leq \delta_1(T)$.

By the inverse mapping of (7), we consider $\eta = \eta(t, s, x, y)$ as a function of $(t, s, x, y) \in \boldsymbol{R}^2 \times \boldsymbol{R}^{2n}$. The curve

$$\gamma_y^x ; \tau \to x(\tau, s, y, \eta(t, s, x, y)) \tag{8}$$

is the unique classical orbit starting y at $t=s$ and arriving at x at $t=t$.

The classical action along the curve γ_y^x is

$$S(t, s, x, y) = \int_s^t L(\sigma, x(\sigma), \dot{x}(\sigma)) d\sigma, \tag{9}$$

where the integral is taken along the curve γ_y^x. The parametrix $e(\lambda, t, s, x, y)$ is of the form

$$e(\lambda, t, s, x, y) = \left(\frac{\lambda}{2\pi(t-s)}\right)^{n/2} a(\lambda, t, s, x, y) e^{\lambda S(t,s,x,y)}, \tag{10}$$

where

$$a(\lambda, t, s, x, y) = \sum_{j=0}^{N} \lambda^{-j} a_j(t, s, x, y) \tag{11}$$

with any $N \geq 0$. Each $a_j(t, s, x, y)$ is determined by the transport equation

$$\begin{aligned} &\frac{\partial}{\partial t} a_j + \sum_k \frac{\partial}{\partial x_k} S(t, s, x, y) \frac{\partial a_j}{\partial x_k} \\ &\quad + \frac{1}{2}\left(\Delta S(t, s, x, y) - \frac{n}{t-s}\right) a_j + \frac{1}{2}\Delta a_{j-1} = 0 \end{aligned} \tag{12}$$

for $0 \leq j \leq N$. Here we put $a_{-1} \equiv 0$. The initial condition is

$$a_0(s, s, y, y) = 1, \quad a_j(s, s, y, y) = 0 \qquad \text{for } 1 \leq j \leq N. \tag{13}$$

§ 3. **Main results.** With the function $e(\lambda, t, s, x, y)$ of (10), we can define a linear integral transformation $E(\lambda, t, s)$ as follows: For any $\varphi \in \mathcal{S}(\boldsymbol{R}^n)$,

$$E(\lambda, t, s)\varphi(x)=\int_{R^n} e(\lambda, t, s, x, y)\varphi(y)dy. \tag{14}$$

This is an oscillatory integral transformation. (See for instances Fujiwara [5] and Asada-Fujiwara [2].) Let $\|\ \|$ denote the usual norm in $L^2(\boldsymbol{R}^n)$.

Proposition 2. (i) *There exists a positive constant* γ_1 *such that*

$$\|E(\lambda, t, s)\varphi\| \leq \gamma_1 \|\varphi\| \tag{15}$$

for any $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ *and* $|t|, |s| \leq T$ *and* $|t-s| \leq \delta_1(T)$.

(ii)

$$s\text{-}\lim_{t\to s} E(\lambda, t, s)\varphi=\varphi \tag{16}$$

for any $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. *Here s-lim means the strong limit in* $L^2(\boldsymbol{R}^n)$.

Let $\varDelta$; $s=t_0<t_1<t_2<\cdots<t_L=t$ be an arbitrary subdivision of the interval $[s, t]$. We put

$$\delta(\varDelta)=\max_{1\leq j\leq N} |t_j-t_{j-1}|. \tag{17}$$

Define

$$E_\varDelta(\lambda, t, s)=E(\lambda, t, t_{L-1})E(\lambda, t_{L-1}, t_{L-2})\cdots E(\lambda, t_1, s) \tag{18}$$

and

$$E_\varDelta(\lambda, s, t)=E(\lambda, s, t_1)E(\lambda, t_1, t_2)\cdots E(\lambda, t_{L-1}, t), \tag{19}$$

for subdivision $\varDelta$.

We discuss the limit of $E_\varDelta(\lambda, t, s)$ and $E_\varDelta(\lambda, s, t)$ when $\delta(\varDelta)\to 0$.

Theorem 1. *Let s and t be arbitrarily fixed. Assume that* $V(t, x)$ *satisfies assumptions* (V-I) *and* (V-II). *Let* N *in* (11) *be larger or equal to* 0. *Then there exist two bounded linear operators* $U(\lambda, t, s)$ *and* $U(\lambda, s, t)$ *of* $L^2(\boldsymbol{R}^n)$ *such that*

$$\lim_{\delta(\varDelta)\to 0} \| U(\lambda, t, s)-E_\varDelta(\lambda, t, s)\|=0, \tag{20}$$

$$\lim_{\delta(\varDelta)\to 0} \| U(\lambda, s, t)-E_\varDelta(\lambda, s, t)\|=0. \tag{21}$$

More precisely, there is a positive constant γ_2 *such that*

$$\| U(\lambda, s, t)-E_\varDelta(\lambda, s, t)\| \leq \gamma_2 |\lambda|^{-N-1} |t-s| \cdot \delta(\varDelta)^{N+1} e^{\gamma_2 |t-s||\lambda|^{-N-1}} \tag{22}$$

and

$$\| U(\lambda, t, s)-E_\varDelta(\lambda, t, s)\| \leq \gamma_2 |\lambda|^{-N-1} |t-s|\, \delta(\varDelta)^{N+1} e^{\gamma_2 |t-s||\lambda|^{-N-1}}. \tag{23}$$

γ_2 *is independent of* $\varDelta$ *and* λ *provided that* $|\lambda|\geq 1$.

Remark 1. Operators $U(\lambda, t, s)$ and $U(\lambda, s, t)$ are independent of particular choice of N in (11) provided that $N\geq 0$. (22) and (23) imply that convergence of (20) and (21) is faster if N is larger.

Remark 2. Theorem 1 holds even if $V(t, x)$ satisfies weaker assumptions below:

(V-I)′ $V(t, x)$ is a real valued function which is measurable in $(t, x) \in \boldsymbol{R}\times\boldsymbol{R}^n$ and infinitely differentiable in x if $t \in \boldsymbol{R}$ is fixed.

(V-II)′ For any multi-index α with length $|\alpha|\geq 2$, the non negative measurable function of t defined by

$$M_\alpha(t)=\sup_{x\in R^n} \left|\left(\frac{\partial}{\partial x}\right)^\alpha V(t, x)\right| + \sup_{|x|\leq 1} |V(t, x)|$$

is essentially bounded on every compact subset of $\boldsymbol{R}^1$.

Theorem 2. *Under the same assumption as that of Theorem* 1, $U(\lambda, t, s)^{-1} = U(\lambda, s, t)$. $\{U(\lambda, t, s)\}_{(t,s)\in \boldsymbol{R}^2}$ *is a family of unitary operators satisfying the following properties*:

(i) $U(\lambda, s, s) = I$.

(ii) $U(\lambda, t, s)$ *is strongly continuous in* t *and* s.

(iii) $U(\lambda, t, s) = U(\lambda, t, s_1)U(\lambda, s_1, s)$ *for any* $t, s_1, s \in \boldsymbol{R}$.

(iv) *For any* $\varphi \in \mathcal{S}(\boldsymbol{R}^n)$, *let* $u(t, x) = U(\lambda, t, s)\varphi(x)$. *Then* $u(t, x)$ *satisfies*

$$\frac{\partial}{\lambda \partial t} u(t, x) + A(\lambda, t)u(t, x) = 0 \qquad \text{in } (t, x) \in \boldsymbol{R}^{n+1} \tag{24}$$

and

$$u(s, x) = \varphi(x),$$

where $A(\lambda, t)$ *is the minimal closed extension of* $\left(\frac{1}{\lambda}\frac{\partial}{\partial x_j}\right)^2 + V(t, x)$ *restricted to* $\mathcal{S}(\boldsymbol{R}^n)$.

§ 4. Sketch of proof. Proposition 2 of § 3 can be proved by L^2-theory of oscillatory integral transformations and stationary phase method. (As to L^2-theory of oscillatory integral transformations see [5], [2].)

In order to prove Theorems 1 and 2, we need the followings:

Proposition 3. *For any fixed* $T > 0$, *there is a positive constant* $\delta_3(T) > 0$ *such that* $E(\lambda, t, s)$ *has its bounded inverse if* $|t|, |s| \leq T$ *and* $|t - s| \leq \delta_2(T)$.

Proposition 4. *Let* T *and* $\delta_2(T)$ *be as above. If* $|t| \leq T, |s| \leq T$, *and* $|t - s| \leq \delta_2(T)$, *then we have the following estimates*:

(i) $\|E(\lambda, t, s)^* - E(\lambda, t, s)^{-1}\| \leq \gamma_3 |t-s|^{N+2} |\lambda|^{-N-1}$,

(ii) $\|E(\lambda, t, s)\| \leq \exp(\gamma_3 |\lambda|^{-N-1} |t-s|^{N+2})$,

(iii) $\|E(\lambda, t, s) - E(\lambda, t, s_1)E(\lambda, s_1, s)\| \leq \gamma_3 |\lambda|^{-N-1}(|t-s_1|^{N+2} + |s-s_1|^{N+2})$,

(iv) $\|E(\lambda, t, s)E(\lambda, s, t) - I\| \leq \gamma_3 |\lambda|^{-N-1} |t-s|^{N+2}$,

where γ_3 *is a positive constant independent of* t, s, s_1 *and* λ *provided that* $|\lambda| \geq 1$.

References

[1] Albeverio, A., and Krohn, H.: Mathematical theory of Feynman path integrals. Springer Lecture Notes in Mathematics, Springer, p. 523 (1976).

[2] Asada, K., and Fujiwara, D.: On some oscillatory integral transformations in $L^2(\boldsymbol{R}^n)$ (to appear in Japanese Journal of Mathematics).

[3] Feynman, R.: Space time approach to non relativistic quantum mechanics. Rev. Modern Physics, **20**, 367–387 (1948).

[4] Fujiwara, D.: Fundamental solution of partial differential operators of

Schrödinger's type. I, II. Proc. Japan Acad., **50**, 566–569; 699–701 (1974).

[5] Fujiwara, D: On the boundedness of integral transformations with rapidly oscillatory kernel. Proc. Japan Acad., **51**, 96–99 (1975).

[6] Itô, K.: Generalized uniform complex measures in the Hilbertian metric space with their application to the Feynman path integral. Proc. 5th Berkeley Symposium on Mathematical Statistics and Probability, Univ. California Press, Berkeley, **2**, 145–161 (1967).

[7] Keller, J. B.-McLaughlin: The Feynman integral. AMS Monthly, **82**, 451–465 (1975).

[8] Nelson, E.: Feynman integrals and Schrödinger equation. J. Math. Physics, **5**, 332–343 (1964).